Bursting Dynamics of the 3D Euler Equations in Cylindrical Domains

François Golse, Alex Mahalov, Basil Nicolaenko

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François Golse * †
Ecole Polytechnique, CMLS
91128 Palaiseau Cedex, France

Alex Mahalov ‡ and Basil Nicolaenko §
Department of Mathematics and Statistics
Arizona State University
Tempe, AZ 85287-1804, USA

Abstract

A class of three-dimensional initial data characterized by uniformly large vorticity is considered for the 3D incompressible Euler equations in bounded cylindrical domains. The fast singular oscillating limits of the 3D Euler equations are investigated for parametrically resonant cylinders. Resonances of fast oscillating swirling Beltrami waves deplete the Euler nonlinearity. These waves are exact solutions of the 3D Euler equations. We construct the 3D resonant Euler systems; the latter are countable uncoupled and coupled $SO(3;\mathbf{C})$ and $SO(3;\mathbf{R})$ rigid body systems. They conserve both energy and helicity. The 3D resonant Euler systems are vested with bursting dynamics, where the ratio of the enstrophy at time $t = t^*$ to the enstrophy at $t = 0$ of some remarkable orbits becomes very large for very small times t^*; similarly for higher norms $\mathbf{H}^s$, $s \geq 2$. These orbits are topologically close to homoclinic cycles. For the time intervals where $\mathbf{H}^s$ norms, $s \geq 7/2$ of the limit resonant orbits do not blow up, we prove that the full 3D Euler equations possess smooth solutions close to the resonant orbits uniformly in strong norms.

Key-Words: Incompressible Euler Equations, Rotating Fluids, Rigid Body Dynamics, Enstrophy Bursts

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*golse@math.polytechnique.fr
†and Laboratoire J.-L. Lions, Université Paris Diderot-Paris 7
‡mahalov@asu.edu
§byn@stokes.la.asu.edu

1 Introduction

The issues of blowup of smooth solutions and finite time singularities of the vorticity field for 3D incompressible Euler equations are still a major open problem. The Cauchy problem in 3D bounded axisymmetric cylindrical domains is attracting considerable attention: with bounded, smooth, *non*-axisymmetric 3D initial data, under the constraints of conservation of *bounded* energy, can the vorticity field blow up in *finite* time? Outstanding numerical claims for this have recently been disproven [Ke], [Hou1], [Hou2]. The classical analytical criterion of Beale-Kato-Majda [B-K-M] for non-blow up in finite time requires the time integrability of the L^∞ norm of the vorticity. DiPerna and Lions [Li] have given examples of global weak solutions of the 3D Euler equations which are smooth (hence unique) if the initial conditions are smooth (specifically in $\mathbf{W}^{1,p}(D),\ p > 1$). However, these flows are really 2-Dimensional in x_1, x_2, 3-components flows, independent from the third coordinate x_3. Their examples [DiPe-Li] show that solutions (even smooth ones) of the 3D Euler equations cannot be estimated in $\mathbf{W}^{1,p}$ for $1 < p < \infty$ on any time interval $(0,\, T)$ if the initial data are only assumed to be bounded in $\mathbf{W}^{1,p}$. Classical local existence theorems in 3D bounded or periodic domains by Kato [Ka], Bourguignon-Brézis [Bou-Br] and Yudovich [Yu1], [Yu2] require some minimal smoothness for the initial conditions (IC), e.g., in $\mathbf{H}^s(D),\ s > \frac{5}{2}$.

The classical formulation for the Euler equations is

$$\partial_t \mathbf{V} + (\mathbf{V}\cdot\nabla)\mathbf{V} = -\nabla p,\ \nabla\cdot\mathbf{V} = 0, \tag{1.1}$$

$$\mathbf{V}\cdot\mathbf{N} = 0 \text{ on } \partial D, \tag{1.2}$$

where ∂D is the boundary of a bounded, connected domain D, $\mathbf{N}$ the normal to ∂D, $\mathbf{V}(t,y) = (V_1, V_2, V_3)$ the velocity field, $y = (y_1, y_2, y_3)$, and p is the pressure.

The equivalent Lamé form [Ar-Khe]

$$\partial_t \mathbf{V} + curl\mathbf{V}\times\mathbf{V} + \nabla\left(p + \frac{1}{2}|\mathbf{V}|^2\right) = 0, \tag{1.3}$$

$$\nabla\cdot\mathbf{V} = 0, \tag{1.4}$$

$$\partial_t \boldsymbol{\omega} + curl(\boldsymbol{\omega}\times\mathbf{V}) = 0, \tag{1.5a}$$

$$\boldsymbol{\omega} = curl\mathbf{V}, \tag{1.5b}$$

implies conservation of Energy:

$$E(t) = \frac{1}{2}\int_D |\mathbf{V}(t,y)|^2\, dy. \tag{1.6}$$

The helicity $Hel(t)$ [Ar-Khe], [Mof], is conserved:

$$Hel(t) = \int_D \mathbf{V}\cdot\boldsymbol{\omega}\, dy, \tag{1.7}$$

for $D = \mathbf{R}^3$ and when D is a periodic lattice. Helicity is also conserved for cylindrical domains, provided that $\boldsymbol{\omega}\cdot\mathbf{N} = 0$ on the cylinder's lateral boundary at $t = 0$ (see [M-N-B-G]).

From the theoretical point of view, the principal difficulty in the analysis of 3D Euler equations is due to the presence of the vortex stretching term $(\boldsymbol{\omega} \cdot \nabla)\mathbf{V}$ in the vorticity equation (1.5a). The equations (1.3) and (1.5a) are equivalent to:

$$\partial_t \boldsymbol{\omega} + [\boldsymbol{\omega}, \mathbf{V}] = 0, \tag{1.8}$$

where $[a, b] =$ curl $(\mathbf{a} \times \mathbf{b})$ is the commutator in the infinite dimensional Lie algebra of divergence-free vector fields [Ar-Khe]. This point of view has led to celebrated developments in Topological Methods in Hydrodynamics [Ar-Khe], [Mof]. The striking analogy between the Euler equations for hydrodynamics and the Euler equations for a rigid body (the latter associated to the Lie Algebra of the Lie group $SO(3, \mathbf{R})$) had already been pointed out by Moreau [Mor1]; Moreau was the first to demonstrate conservation of Helicity (1961) [Mor2]. This has led to extensive speculations to what extent/in what cases are the solutions of the 3D Euler equations "close" to those of coupled 3D rigid body equations in some asymptotic sense. Recall that the Euler equations for a rigid body in $\mathbf{R}^3$ is:

$$\mathbf{m}_t + \boldsymbol{\omega} \times \mathbf{m} = 0, \quad \mathbf{m} = A\boldsymbol{\omega}, \tag{1.9a}$$

$$\mathbf{m}_t + [\boldsymbol{\omega}, \mathbf{m}] = 0, \tag{1.9b}$$

where $\mathbf{m}$ is the vector of angular momentum relative to the body, $\boldsymbol{\omega}$ the angular velocity in the body and A the inertia operator [Ar1], [Ar-Khe].

The Russian school of Gledzer, Dolzhansky, Obukhov [G-D-O] and Vishik [Vish] has extensively investigated dynamical systems of hydrodynamic type and their applications. They have considered hydrodynamical models built upon generalized rigid body systems in $SO(n, \mathbf{R})$, following Manakhov [Man]. Inspired by turbulence physics, they have investigated "shell" dynamical systems modeling turbulence cascades; albeit such systems are flawed as they only preserve energy, not helicity. To address this, they have constructed and studied in depth n-dimensional dynamical systems with quadratic homogeneous nonlinearities and **two** quadratic first integrals F_1, F_2. Such systems can be written using sums of Poisson brackets:

$$\frac{dx^{i_1}}{dt} = \frac{1}{2} \sum_{i_2,\ldots,i_n} \epsilon^{i_1 i_2 \ldots i_n} p_{i_4 \ldots i_n} \left(\frac{\partial F_1}{\partial x^{i_2}} \frac{\partial F_2}{\partial x^{i_3}} - \frac{\partial F_1}{\partial x^{i_3}} \frac{\partial F_2}{\partial x^{i_2}} \right), \tag{1.10}$$

where constants $p_{i_4 \ldots i_n}$ are antisymmetric in $i_4, \ldots, i_n$.

A simple version of such a quadratic hydrodynamic system was introduced by Gledzer [Gl1] in 1973. A deep open issue of the work by the Gledzer-Obukhov school is whether there exist indeed classes of I.C. for the 3D Cauchy Euler problem (1.1) for which solutions are actually asymptotically close in strong norm, on arbitrary large time intervals to solutions of such hydrodynamic systems, with conservation of both energy and helicity. Another unresolved issue is the blowup or global regularity for the "enstrophy" of such systems when their dimension $n \to \infty$.

This article reviews some current new results of a research program in the spirit of the Gledzer-Obukhov school; this program builds-up on the results of [M-N-B-G] for 3D Euler in bounded cylindrical domains. Following the original approach of [B-M-N1]-[B-M-N4] in periodic domains, [M-N-B-G] prove the non blowup of the 3D incompressible Euler equations for a class of three-dimensional initial data characterized by uniformly large vorticity in bounded cylindrical domains. There are no conditional assumptions on the properties of solutions at later times, nor are the global solutions close to some 2D manifold. The initial vortex stretching is large. The approach of proving regularity is based on investigation of fast singular oscillating limits and nonlinear averaging methods in the context of almost periodic functions [Bo-Mi], [Bes], [Cor]. Harmonic analysis tools based on curl eigenfunctions and eigenvalues are crucial. One establishes the global regularity of the 3D limit resonant Euler equations without any restriction on the size of 3D initial data. The resonant Euler equations are characterized by a depleted nonlinearity. After establishing strong convergence to the limit resonant equations, one bootstraps this into the regularity on arbitrary large time intervals of the solutions of 3D Euler Equations with weakly aligned uniformly large vorticity at $t = 0$. [M-N-B-G] theorems hold for generic cylindrical domains, for a set of height/radius ratios of full Lebesgue measure. For such cylinders, the 3D limit resonant Euler equations are restricted to two-wave resonances of the vorticity waves and are vested with an infinite countable number of new conservation laws. The latter are adiabatic invariants for the original 3D Euler equations.

Three-wave resonances exist for a nonempty countable set of h/R (h height, R radius of the cylinder) and moreover accumulate in the limit of vanishingly small vertical (axial) scales. This is akin to Arnold tongues [Ar2] for the Mathieu-Hill equations and raises nontrivial issues of possible singularities/lack thereof for dynamics ruled by infinitely many resonant triads at vanishingly small axial scales. In such a context, the 3D resonant Euler equations do conserve the energy and helicity of the field.

In this review, we consider cylindrical domains with parametric resonances in h/R and investigate in depth the structure and dynamics of 3D resonant Euler systems. These parametric resonances in h/R are proven to be nonempty. Solutions to Euler equations with uniformly large initial vorticity are expanded along a full *complete* basis of elementary swirling waves ($\mathbf{T}^2$ in time). Each such quasiperiodic, dispersive vorticity wave is a quasiperiodic Beltrami flow; these are exact solutions of 3D Euler equations with vorticity parallel to velocity. There are no Galerkin-like truncations in the decomposition of the full 3D Euler field. The Euler equations, restricted to resonant triplets of these dispersive Beltrami waves, determine the "resonant Euler systems". The basic "building block" of these (a priori ∞-dimensional) systems are proven to be $SO(3;\mathbf{C})$ and $SO(3;\mathbf{R})$ rigid body systems:

$$\begin{aligned}\dot{U}_k + (\lambda_m - \lambda_n)U_m U_n = 0\\ \dot{U}_m + (\lambda_n - \lambda_k)U_n U_k = 0\\ \dot{U}_n + (\lambda_k - \lambda_m)U_k U_m = 0\end{aligned} \tag{1.11}$$

These λ's are *eigenvalues of the curl operator* in the cylinder, $curl\mathbf{\Phi}_n^{\pm} = \pm\lambda_n\mathbf{\Phi}_n^{\pm}$; the curl eigenfunctions are steady elementary Beltrami flows, and the dispersive Beltrami waves oscillate with the frequencies $\pm\frac{h}{2\pi\epsilon}\frac{n_3}{\lambda_n}$, n_3 vertical wave number (vertical shear), $0 < \epsilon < 1$. Physicists [Ch-Ch-Ey-H] have computationally demonstrated the physical impact of the polarization of Beltrami modes $\mathbf{\Phi}^{\pm}$ on intermittency in the joint cascade of energy and helicity in turbulence.

Another "building block" for resonant Euler systems is a pair of $SO(3;\mathbf{C})$ or $SO(3;\mathbf{R})$ rigid bodies coupled via a common principal axis of inertia/moment of inertia:

$$\dot{a}_k = (\lambda_m - \lambda_n)\Gamma a_m a_n \tag{1.12a}$$
$$\dot{a}_m = (\lambda_n - \lambda_k)\Gamma a_n a_k \tag{1.12b}$$
$$\dot{a}_n = (\lambda_k - \lambda_m)\Gamma a_k a_m + (\lambda_{\tilde{k}} - \lambda_{\tilde{m}})\tilde{\Gamma} a_{\tilde{k}} a_{\tilde{m}} \tag{1.12c}$$
$$\dot{a}_{\tilde{m}} = (\lambda_n - \lambda_{\tilde{k}})\tilde{\Gamma} a_n a_{\tilde{k}} \tag{1.12d}$$
$$\dot{a}_{\tilde{k}} = (\lambda_{\tilde{m}} - \lambda_n)\tilde{\Gamma} a_{\tilde{m}} a_n, \tag{1.12e}$$

where Γ and $\tilde{\Gamma}$ are parameters in $\mathbf{R}$ defined in Theorem 4.10. Both resonant systems (1.11) and (1.12) conserve energy and helicity. We prove that the dynamics of these resonant systems admit equivariant families of homoclinic cycles connecting hyperbolic critical points. We demonstrate bursting dynamics: the ratio

$$||\mathbf{u}(t)||^2_{H^s}/||\mathbf{u}(0)||^2_{H^s},\ s \geq 1$$

can burst arbitrarily large on arbitrarily small times, for properly chosen parametric domain resonances h/R. Here

$$||\mathbf{u}(t)||^2_{H^s} = \sum_n (\lambda_n)^{2s}|u_n(t)|^2\,. \tag{1.13}$$

The case $s = 1$ is the enstrophy. The "bursting" orbits are topologically close to the homoclinic cycles.

Are such dynamics for the resonant systems relevant to the full 3D Euler equations (1.1)-(1.8)? The answer lies in the following crucial "shadowing" Theorem 2.10. Given the same initial conditions, given the maximal time interval $0 \leq t < T_m$ where the resonant orbits of the resonant Euler equations do **not** blow up, then the *strong* norm $\mathbf{H}^s$ of the difference between the exact

Euler orbit and the resonant orbit is uniformly small on $0 \leq t < T_m$, provided that the vorticity of the I.C. is large enough. Paradoxically, the larger the vortex streching of the I.C., the better the uniform approximation. This deep result is based on cancellation of fast oscillations in *strong* norms, in the context of almost periodic functions of time with values in Banach spaces (Section 4 of [M-N-B-G]). It includes uniform approximation in the spaces $\mathbf{H}^s,\ s > 5/2$. For instance, given a quasiperiodic orbit on some time torus $\mathbf{T}^l$ for the resonant Euler systems, the exact solutions to the Euler equations will remain ϵ-close to the resonant quasiperiodic orbit on a time interval $0 \leq t \leq \max T_i$, $1 \leq i \leq l$, T_i elementary periods, for large enough initial vorticity. If orbits of the resonant Euler systems admit bursting dynamics in the strong norms $\mathbf{H}^s,\ s \geq 7/2$, so do *some exact solutions of the full 3D Euler equations*, for properly chosen parametrically resonant cylinders.

2 Vorticity waves and resonances of elementary swirling flows

We study initial value problem for the three-dimensional Euler equations with initial data characterized by uniformly large vorticity:

$$\partial_t \mathbf{V} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\nabla p,\ \nabla \cdot \mathbf{V} = 0, \tag{2.1}$$

$$\mathbf{V}(t,y)|_{t=0} = \mathbf{V}(0) = \tilde{\mathbf{V}}_0(y) + \frac{\Omega}{2}\mathbf{e}_3 \times y \tag{2.2}$$

where $y = (y_1, y_2, y_3)$, $\mathbf{V}(t,y) = (V_1, V_2, V_3)$ is the velocity field and p is the pressure. In Eqs. (1.1) $\mathbf{e}_3$ denotes the vertical unit vector and Ω is a constant parameter. The field $\tilde{\mathbf{V}}_0(y)$ depends on three variables y_1, y_2 and y_3. Since $curl(\frac{\Omega}{2}\mathbf{e}_3 \times y) = \Omega\mathbf{e}_3$, the vorticity vector at initial time $t = 0$ is

$$curl\mathbf{V}(0,y) = curl\tilde{\mathbf{V}}_0(y) + \Omega\mathbf{e}_3, \tag{2.3}$$

and the initial vorticity has a large component weakly aligned along $\mathbf{e}_3$, when $\Omega >> 1$. These are fully three-dimensional large initial data with large initial 3D vortex stretching. We denote by $\mathbf{H}^s_\sigma$ the usual Sobolev space of solenoidal vector fields.

The base flow

$$\mathbf{V}_s(y) = \frac{\Omega}{2}\mathbf{e}_3 \times y,\ curl\mathbf{V}_s(y) = \Omega\mathbf{e}_3 \tag{2.4}$$

is called a steady swirling flow and is a steady state solution (1.1)-(1.4), as $curl(\Omega\mathbf{e}_3 \times \mathbf{V}_s(y)) = 0$. In (2.2) and (2.3), we consider I.C. which are an arbitrary ***(not small)*** perturbation of the base swirling flow $\mathbf{V}_s(y)$ and introduce

$$\mathbf{V}(t,y) = \frac{\Omega}{2}\mathbf{e}_3 \times y + \tilde{\mathbf{V}}(t,y), \tag{2.5}$$

$$curl\mathbf{V}(t,y) = \Omega\mathbf{e}_3 + curl\tilde{\mathbf{V}}(t,y), \tag{2.6}$$

$$\partial_t\tilde{\mathbf{V}} + curl\tilde{\mathbf{V}} \times \tilde{\mathbf{V}} + \Omega\mathbf{e}_3 \times \tilde{\mathbf{V}} + curl\tilde{\mathbf{V}} \times \mathbf{V}_s(y) + \nabla p' = 0,\ \nabla\cdot\tilde{\mathbf{V}} = 0, \tag{2.7}$$

$$\tilde{\mathbf{V}}(t,y)|_{t=0} = \tilde{\mathbf{V}}_0(y). \tag{2.8}$$

Eqs. (2.1) and (2.7) are studied in cylindrical domains

$$\mathbf{C} = \{(y_1,y_2,y_3) \in \mathbf{R}^3 : 0 < y_3 < 2\pi/\alpha,\ y_1^2 + y_2^2 < R^2\} \tag{2.9}$$

where α and R are positive real numbers. If h is the height of the cylinder, $\alpha = 2\pi/h$. Let

$$\Gamma = \{(y_1,y_2,y_3) \in \mathbf{R}^3 : 0 < y_3 < 2\pi/\alpha,\ y_1^2 + y_2^2 = R^2\}. \tag{2.10}$$

Without loss of generality, we can assume that $R = 1$. Eqs. (2.1) are considered with periodic boundary conditions in y_3

$$\mathbf{V}(y_1,y_2,y_3) = \mathbf{V}(y_1,y_2,y_3 + 2\pi/\alpha) \tag{2.11}$$

and vanishing normal component of velocity on Γ

$$\mathbf{V}\cdot\mathbf{N} = \tilde{\mathbf{V}}\cdot\mathbf{N} = 0 \text{ on } \Gamma; \tag{2.12}$$

where $\mathbf{N}$ is the normal vector to Γ. From the invariance of 3D Euler equations under the symmetry $y_3 \to -y_3$, $V_1 \to V_1$, $V_2 \to V_2$, $V_3 \to -V_3$, all results in this article extend to cylindrical domains bounded by two horizontal plates. Then the boundary conditions in the vertical direction are zero flux on the vertical boundaries (zero vertical velocity on the plates). One only needs to restrict vector fields to be even in y_3 for V_1, V_2 and odd in y_3 for V_3, and double the cylindrical domain to $-h \le y_3 \le +h$.

We choose $\tilde{\mathbf{V}}_0(y)$ in $\mathbf{H}^s(\mathbf{C})$, $s > 5/2$. In [M-N-B-G], for the case of "non-resonant cylinders", that is, non-resonant $\alpha = 2\pi/h$, we have established regularity for arbitrarily large finite times for the 3D Euler solutions for Ω large, but *finite*. Our solutions are not close in any sense to those of the 2D or "quasi 2D" Euler and they are characterized by fast oscillations in the $\mathbf{e}_3$ direction, together with a large vortex stretching term

$$\boldsymbol{\omega}(t,y)\cdot\nabla\mathbf{V}(t,y) = \omega_1\frac{\partial\mathbf{V}}{\partial y_1} + \omega_2\frac{\partial\mathbf{V}}{\partial y_2} + \omega_3\frac{\partial\mathbf{V}}{\partial y_3},\quad t \ge 0$$

with leading component $\left|\Omega\frac{\partial}{\partial y_3}\mathbf{V}(t,y)\right| \gg 1$. There are no assumptions on oscillations in y_1, y_2 for our solutions (nor for the initial condition $\tilde{\mathbf{V}}_0(y)$).

Our approach is entirely based on sturying fast singular oscillating limits of Eqs. (1.1)-(1.5a), nonlinear averaging and cancelation of oscillations in the nonlinear interactions for the vorticity field for large Ω. This has been developed in [B-M-N2], [B-M-N3], and [B-M-N4] for the cases of periodic lattice domains and the infinite space $\mathbf{R}^3$.

It is well known that fully three-dimensional initial conditions with uniformly large vorticity excite fast Poincaré vorticity waves [B-M-N2], [B-M-N3], [B-M-N4], [Poi]. Since individual Poincaré wave modes are related to the eigenfunctions of the curl operator, they are exact time-dependent solutions of the full nonlinear 3D Euler equations. Of course, their linear superposition does not preserve this property. Expanding solutions of (2.1)-(2.8) along such vorticity waves demonstrates potential nonlinear resonances of such waves. First recall spectral properties of the *curl* operator in bounded, connected domains:

Proposition 2.1 ([M-N-B-G]) *The curl operator admits a self-adjoint extension under the zero flux boundary conditions, with a discrete real spectrum $\lambda_n = \pm|\lambda_n|$, $|\lambda_n| > 0$ for every n and $|\lambda_n| \to +\infty$ as $|n| \to \infty$. The corresponding eigenfunctions $\mathbf{\Phi}_n^{\pm}$*

$$curl\mathbf{\Phi}_n^{\pm} = \pm|\lambda_n|\mathbf{\Phi}_n^{\pm} \tag{2.13}$$

are complete in the space

$$\boldsymbol{J}^0 = \left\{ \boldsymbol{U} \in L^2(D) : \nabla \cdot \mathbf{U} = 0 \ and\ \mathbf{U} \cdot \mathbf{N}|_{\partial D} = 0 \ and\ \int_o^h \mathbf{U}\, dz = 0 \right\}. \tag{2.14}$$

Remark 2.2 *In cylindrical domains, with cylindrical coordinates (r, θ, z), the eigenfunctions admit the representation:*

$$\mathbf{\Phi}_{n_1,n_2,n_3} = (\Phi_{r,n_1,n_2,n_3}(r), \Phi_{\theta,n_1,n_2,n_3}(r), \Phi_{z,n_1,n_2,n_3}(r))\, e^{in_2\theta} e^{i\alpha n_3 z}, \tag{2.15}$$

with $n_2 = 0, \pm1, \pm2, \ldots$, $n_3 = \pm1, \pm2, \ldots$ and $n_1 = 0, 1, 2, \ldots$. Here n_1 indexes the eigenvalues of the equivalent Sturm-Liouville problem in the radial coordinates, and $n = (n_1, n_2, n_3)$. See [M-N-B-G] *for technical details. From now on, we use the generic variable z for any vertical (axial) coordinate y_3 or x_3. For $n_3 = 0$ (vertical averaging along the axis of the cylinder), 2-Dimensional, 3-component solenoidal fields must be expanded along a complete basis for fields derived from 2D stream functions:*

$$\begin{aligned} \mathbf{\Phi}_n &= \big((curl(\phi_n \mathbf{e}_3),\, \phi_n \mathbf{e}_3)\big), \quad \phi_n = \phi_n(r, \theta), \\ -\triangle\phi_n &= \mu_n \phi_n,\ \phi_n|_{\partial\Gamma} = 0, \quad and \\ curl\mathbf{\Phi}_n &= \big((curl(\phi_n \mathbf{e}_3),\, \mu_n \phi_n \mathbf{e}_3)\big). \end{aligned}$$

Here $\big((\mathbf{a},\, b\mathbf{e}_3)\big)$ denotes a 3-component vector whose horizontal projection is $\mathbf{a}$ and vertical projection is $b\mathbf{e}_3$.

Let us explicit elementary swirling wave flows which are *exact* solutions to (2.1) and (2.7):

Lemma 2.3 *For every $n = (n_1, n_2, n_3)$, the following quasiperiodic ($\mathbf{T}^2$ in time) solenoidal fields are exact solution of the full 3D nonlinear Euler equations (2.1):*

$$\mathbf{V}(t,y) = \frac{\Omega}{2}\mathbf{e}_3 \times y + \exp(\frac{\Omega}{2}\mathbf{J}t)\mathbf{\Phi}_n(\exp(-\frac{\Omega}{2}\mathbf{J}t)y)\exp(\pm i\frac{n_3}{|\lambda_n|}\alpha\Omega t), \quad (2.16)$$

n_3 is the vertical wave number of $\mathbf{\Phi}_n$ and $\exp(\frac{\Omega}{2}\mathbf{J}t)$ the unitary group of rigid body rotations:

$$\mathbf{J} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; e^{\Omega\mathbf{J}t/2} = \begin{pmatrix} \cos(\frac{\Omega t}{2}) & -\sin(\frac{\Omega t}{2}) & 0 \\ \sin(\frac{\Omega t}{2}) & \cos(\frac{\Omega t}{2}) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.17)$$

Remark 2.4 *These fields are exact quasiperiodic, nonaxisymmetric swirling flow solutions of the 3D Euler equations. For $n_3 \neq 0$, their second components*

$$\tilde{\mathbf{V}}(t,y) = \exp(\frac{\Omega}{2}\mathbf{J}t)\mathbf{\Phi}_n(\exp(-\frac{\Omega}{2}\mathbf{J}t)y)\exp(\pm\frac{in_3}{|\lambda_n|}\alpha\Omega t) \quad (2.18)$$

are Beltrami flows ($curl\tilde{\mathbf{V}} \times \tilde{\mathbf{V}} \equiv 0$) exact solutions of (2.7) with $\tilde{\mathbf{V}}(t=0,y) = \mathbf{\Phi}_n(y)$.

$\tilde{\mathbf{V}}(t,y)$ in Eq. (2.18) are dispersive waves with frequencies $\frac{\Omega}{2}$ and $\frac{n_3\alpha}{|\lambda_n|}\Omega$, where $\alpha = \frac{2\pi}{h}$. Moreover, each $\tilde{\mathbf{V}}(t,y)$ is a *traveling wave* along the cylinder's axis, since it contains the factor

$$\exp\left(i\alpha n_3(\pm z \pm \frac{\Omega}{|\lambda_n|}t)\right).$$

Note that n_3 large corresponds to small axial (vertical) scales, albeit $0 \leq \alpha|n_3/\lambda_n| \leq 1$.

Proof of Lemma 2.3. Through the canonical rigid body transformation for both the field $\mathbf{V}(t,y)$ and the space coordinates $y = (y_1, y_2, y_3)$:

$$\mathbf{V}(t,y) = e^{+\Omega\mathbf{J}t/2}\mathbf{U}(t, e^{-\Omega\mathbf{J}t/2}y) + \frac{\Omega}{2}\mathbf{J}y, \; x = e^{-\Omega\mathbf{J}t/2}y, \quad (2.19)$$

the 3D Euler equations (2.1), (2.2) transform into:

$$\partial_t\mathbf{U} + (curl\mathbf{U} + \Omega\mathbf{e}_3) \times \mathbf{U} = -\nabla\left(p - \frac{\Omega^2}{4}(|x_1|^2 + |x_2|^2) + \frac{|\mathbf{U}|^2}{2}\right), \quad (2.20)$$

$$\nabla \cdot \mathbf{U} = 0, \quad \mathbf{U}(t,x)|_{t=0} = \mathbf{U}(0) = \tilde{\mathbf{V}}_0(x), \quad (2.21)$$

For Beltrami flows such that $curl\mathbf{U} \times \mathbf{U} \equiv 0$, these Euler equations (2.20)-(2.21) in a rotating frame reduce to:

$$\partial_t \mathbf{U} + \Omega \mathbf{e}_3 \times \mathbf{U} + \nabla \pi = 0, \ \nabla \cdot \mathbf{U} = 0,$$

which are identical to the Poincaré-Sobolev nonlocal wave equations in the cylinder [M-N-B-G], [Poi], [Sob], [Ar-Khe]:

$$\partial_t \mathbf{\Psi} + \Omega \mathbf{e}_3 \times \mathbf{\Psi} + \nabla \pi = 0, \ \nabla \cdot \mathbf{\Psi} = 0, \tag{2.22}$$

$$\frac{\partial^2}{\partial t^2} curl^2 \mathbf{\Psi} - \mathbf{\Omega}^2 \frac{\partial^2}{\partial x_3^2} \mathbf{\Psi} = 0, \ \mathbf{\Psi} \cdot \mathbf{N}|_{\partial D} = 0. \tag{2.23}$$

It suffices to verify that the Beltrami flows $\mathbf{\Psi}_n(t,x) = \mathbf{\Phi}_n(x) \exp\left(\pm i \frac{\alpha n_3}{|\lambda_n|} \Omega t\right)$, where $\mathbf{\Phi}_n^{\pm}(x)$ and $\pm|\lambda_n|$ are curl eigenfunctions and eigenvalues, are exact solutions to the Poincaré-Sobolev wave equation, in such a rotating frame of reference.

Remark 2.5 *The frequency spectrum of the Poincaré vorticity waves (solutions to* (2.22)*) is exactly* $\pm i \frac{\alpha n_3}{|\lambda_n|} \Omega$*,* $n = (n_1, n_2, n_3)$ *indexing the spectrum of curl. Note that* $n_3 = 0$ *(zero frequency of rotating waves) corresponds to 2-Dimensional, 3-Components solenoidal vector fields.*

We now transform the Cauchy problem for the 3D Euler equations (2.1)-(2.2) into an infinite dimensional nonlinear dynamical system by expanding $\mathbf{V}(t,y)$ along the swirling wave flows (2.16)-(2.18):

$$\mathbf{V}(t,y) = \frac{\Omega}{2} \mathbf{e}_3 \times y \tag{2.24a}$$

$$+ \exp\left(\frac{\Omega}{2} \mathbf{J} t\right) \left\{ \sum_{n=(n_1,n_2,n_3)} \mathbf{u}_n(t) \exp\left(\pm i \frac{\alpha n_3}{|\lambda_n|} \Omega t\right) \mathbf{\Phi}_n \left(\exp\left(-\frac{\Omega}{2} \mathbf{J} t\right) y \right) \right\} \tag{2.24b}$$

$$\mathbf{V}(t=0,y) = \frac{\Omega}{2} \mathbf{e}_3 \times y + \tilde{\mathbf{V}}_0(y) \tag{2.24c}$$

$$\tilde{\mathbf{V}}_0(y) = \sum_{n=(n_1,n_2,n_3)} \mathbf{u}_n(0) \mathbf{\Phi}_n(y), \tag{2.24d}$$

where $\mathbf{\Phi}_n$ denotes the curl eigenfunctions of Proposition 2.1 if $n_3 \neq 0$, and $\mathbf{\Phi}_n = \big((curl(\phi_n \mathbf{e_3}), \, \phi_n \mathbf{e}_3) \big)$ if $n_3 = 0$ (2D case, Remark 2.2).

As we focus on the case where helicity is conserved for (2.1)-(2.2), we consider the class of initial data $\tilde{\mathbf{V}}_0$ such that [M-N-B-G]:

$$curl \tilde{\mathbf{V}}_0 \cdot \mathbf{N} = 0 \text{ on } \Gamma,$$

where Γ is the lateral boundary of the cylinder.

The infinite dimensional dynamical system is then equivalent to the 3D Euler equations (2.1)-(2.2) in the cylinder, with $n = (n_1, n_2, n_3)$ ranging over the whole spectrum of curl, e.g.:

$$\frac{d\mathbf{u}_n}{dt} = - \sum_{\substack{k,m \\ k_3+m_3=n_3 \\ k_2+m_2=n_2}} \exp\left(i \left(\pm \frac{n_3}{|\lambda_n|} \pm \frac{k_3}{|\lambda_k|} \pm \frac{m_3}{|\lambda_m|} \right) \alpha\Omega t \right) \times < curl\mathbf{\Phi}_k \times \mathbf{\Phi}_m, \mathbf{\Phi}_n > \mathbf{u}_k(t)\mathbf{u}_m(t) \tag{2.25}$$

here

$$curl\mathbf{\Phi}_k^{\pm} = \pm\lambda_k \mathbf{\Phi}_k^{\pm} \quad \text{if} \quad k_3 \neq 0,$$
$$curl\mathbf{\Phi}_k = \Big(\big(curl(\phi_k \mathbf{e}_3),\ \mu_k \phi_k \mathbf{e}_3\big)\Big) \quad \text{if} \quad k_3 = 0$$

(2D, 3-components, Remark 2.2), similarly for $m_3 = 0$ and $n_3 = 0$. The inner product $< \, , \, >$ denotes the L^2 complex-valued inner product in D.

This is an infinite dimensional system of coupled equations with quadratic nonlinearities, which conserve both the energy

$$E(t) = \sum_n |\mathbf{u}_n(t)|^2$$

and the helicity

$$Hel(t) = \sum_n \pm|\lambda_n|\, |\mathbf{u}_n^{\pm}(t)|^2.$$

The quadratic nonlinearities split into *resonant terms* where the exponential oscillating phase factor in (2.25) reduces to unity and fast oscillating non-resonant terms ($\Omega >> 1$). The resonant set K is defined in terms of vertical wavenumbers k_3, m_3, n_3 and eigenvalues $\pm\lambda_k$, $\pm\lambda_m$, $\pm\lambda_n$ of *curl*:

$$K = \{\pm\frac{k_3}{\lambda_k} \pm \frac{m_3}{\lambda_m} \pm \frac{n_3}{\lambda_n} = 0, n_3 = k_3 + m_3, n_2 = k_2 + m_2\}. \tag{2.27}$$

Here k_2, m_2, n_2 are azimuthal wavenumbers.

We shall call the "resonant Euler equations" the following ∞-dimensional dynamical system restricted to $(k, m, n) \in K$:

$$\frac{d\mathbf{u}_n}{dt} + \sum_{(k,m,n)\in K} < curl\mathbf{\Phi}_k \times \mathbf{\Phi}_m, \mathbf{\Phi}_n > \mathbf{u}_k\mathbf{u}_m = 0, \tag{2.28a}$$

$$\mathbf{u}_n(0) \equiv < \tilde{\mathbf{V}}_0, \mathbf{\Phi}_n >, \tag{2.28b}$$

here $curl\mathbf{\Phi}_k^{\pm} = \pm\lambda_k \mathbf{\Phi}_k^{\pm}$ if $k_3 \neq 0$, $curl\mathbf{\Phi}_k = \Big(\big(curl(\phi_k \mathbf{e}_3),\ \mu_k \phi_k \mathbf{e}_3\big)\Big)$ if $k_3 = 0$; similarly for $m_3 = 0$ and $n_3 = 0$ (2D components, Remark 2.2). If there are no terms in (2.28a) satisfying the resonance conditions, then there will be some modes for which $\frac{d\mathbf{u}_j}{dt} = 0$.

Lemma 2.6 *The resonant 3D Euler equations (2.28) conserve both energy $E(t)$ and helicity $Hel(t)$. The energy and helicity are identical to that of the full exact 3D Euler equations (2.1)-(2.2).*

The set of resonances K is studied in depth in [M-N-B-G]. To summarize, K splits into:

(i) 0-wave resonances, with $n_3 = k_3 = m_3 = 0$; the corresponding resonant equations are identical to the 2-Dimensional, 3-Components Euler equations, with I.C.

$$\frac{1}{h}\int_0^h \tilde{\mathbf{V}}_0(y_1, y_2, y_3)\, dy_3.$$

(ii) Two-Wave resonances, with $k_3 m_3 n_3 = 0$, but two of them are not null; the corresponding resonant equations (called "catalytic equations") are proven to possess an infinite, countable set of new conservation laws [M-N-B-G].

(iii) Strict three-wave resonances for a subset $K^* \subset K$.

Definition 2.7 *The set K^* of strict 3 wave resonances is:*

$$K^* = \left\{ \pm\frac{k_3}{\lambda_k} \pm \frac{m_3}{\lambda_m} \pm \frac{n_3}{\lambda_n} = 0, k_3 m_3 n_3 \neq 0, n_3 = k_3 + m_3, n_2 = k_2 + m_2 \right\}. \tag{2.29}$$

Note that K^* is parameterized by h/R, since $\alpha = \frac{2\pi}{h}$ parameterizes the eigenvalues $\lambda_n, \lambda_k, \lambda_m$ of the curl operator.

Proposition 2.8 *There exist a countable, non-empty set of parameters $\frac{h}{R}$ for which $K^* \neq \varnothing$.*

Proof. The technical details, together with a more precise statement, are postponed to the proof of Lemma 3.7. Concrete examples of resonant axisymmetric and helical waves are discussed in [Mah] (cf. Figure 2 in the article).

Corollary 2.9 *Let $\int_0^h \tilde{\mathbf{V}}_0(y_1, y_2, y_3)\, dy_3 = 0$, i.e. zero vertical mean for the I.C. $\tilde{\mathbf{V}}_0(y)$ in (2.2), (2.8), (2.24d) and (2.28b). Then the resonant 3D Euler equations are invariant on K^*:*

$$\frac{d\mathbf{u}_n}{dt} + \sum_{(k,m,n)\in K^*} \lambda_k < \mathbf{\Phi}_k \times \mathbf{\Phi}_m, \mathbf{\Phi}_n > \mathbf{u}_k \mathbf{u}_m = 0,\ k_3 m_3 n_3 \neq 0, \tag{2.30a}$$

$$\mathbf{u}_n(0) =< \tilde{\mathbf{V}}_0, \mathbf{\Phi}_n > \tag{2.30b}$$

(where $\tilde{\mathbf{V}}_0$ has spectrum restricted to $n_3 \neq 0$).

Proof. This is an immediate corollary of the "operator splitting" Theorem 3.2 in [M-N-B-G]. ■

We shall call the above dynamical systems the "*strictly resonant Euler system*". This is an ∞-dimensional Riccati system which conserves Energy and Helicity. It corresponds to *nonlinear interactions depleted* on K^*.

How do dynamics of the resonant Euler equations (2.28) or (2.30) approximate *exact* solutions of the Cauchy problem for the full Euler equations in strong norms? This is answered by the following theorem, proven in Section 4 of [M-N-B-G]:

Theorem 2.10 *Consider the initial value problem*

$$\mathbf{V}(t=0,y) = \frac{\Omega}{2}\mathbf{e}_3 \times y + \tilde{\mathbf{V}}_0(y), \ \ \tilde{\mathbf{V}}_0 \in \mathbf{H}^s_\sigma, \ \ s > 7/2$$

for the full 3D Euler equations, with $||\tilde{\mathbf{V}}_0||_{\mathbf{H}^s_\sigma} \leq M^0_s$ *and* $curl\tilde{\mathbf{V}}_0 \cdot \mathbf{N} = 0$ *on* Γ.

- *Let* $\mathbf{V}(t,y) = \frac{\Omega}{2}\mathbf{e}_3 \times y + \tilde{\mathbf{V}}(t,y)$ *denote the solution to the exact Euler equations.*

- *Let* $\mathbf{w}(t,x)$ *denote the solution to the resonant 3D Euler equations with Initial Condition* $\mathbf{w}(0,x) \equiv \mathbf{w}(0,y) = \tilde{\mathbf{V}}_0(y)$.

- *Let* $||\mathbf{w}(t,y)||_{\mathbf{H}^s_\sigma} \leq M_s(T_M, M^0_s)$ *on* $0 \leq t \leq T_M, \ s > 7/2$.

Then, $\forall \epsilon > 0, \ \exists \ \Omega^*(T_M, M^0_s, \epsilon)$ *such that,* $\forall \Omega \geq \Omega^*$*:*

$$\left|\left|\tilde{\mathbf{V}}(t,y) - \exp\left(\frac{\mathbf{J}\Omega t}{2}\right)\left\{\sum_n \mathbf{u}_n(t) e^{-i\frac{n_3}{\lambda_n}\alpha\Omega t}\mathbf{\Phi}_n(e^{-\frac{\mathbf{J}\Omega t}{2}}y)\right\}\right|\right|_{H^\beta} \leq \epsilon$$

on $0 \leq t \leq T_M, \ \forall \beta \geq 1, \ \beta \leq s-2$. *Here* $||\cdot||_{H^\beta}$ *is defined in (1.13).*

The 3D Euler flow preserves the condition $curl\tilde{\mathbf{V}}_0 \cdot \mathbf{N} = 0$ on Γ, that is $curl\mathbf{V}(t,y) \cdot \mathbf{N} = 0$ on Γ, for every $t \geq 0$ [M-N-B-G]. The proof of this "error-shadowing" theorem is delicate, beyond the usual Gronwall differential inequalities and involves estimates of oscillating integrals of almost periodic functions of time with values in Banach spaces. Its importance lies in that solutions of the resonant Euler equations (2.28) and/or (2.30) are uniformly close in strong norms to those of the exact Euler equations (2.1)-(2.2), on any time interval of existence of smooth solutions of the *resonant* system. The infinite dimensional Riccati systems (2.28) and (2.30) are not just hydrodynamic models, but *exact asymptotic limit systems* for $\Omega \gg 1$. This is in contrast to all previous literature on conservative 3D hydrodynamic models, such as in [G-D-O].

3 Strictly resonant Euler systems: the SO(3) case

We investigate the structure and the dynamics of the "strictly resonant Euler systems" (2.30). Recall that the set of 3-wave resonances is:

$$K^* = \Big\{(k,m,n) : \pm\frac{k_3}{\lambda_k} \pm \frac{m_3}{\lambda_m} \pm \frac{n_3}{\lambda_n} = 0, k_3 m_3 n_3 \neq 0, \\ n_3 = k_3 + m_3, n_2 = k_2 + m_2\Big\}. \tag{3.1}$$

From the symmetries of the curl eigenfunctions $\mathbf{\Phi}_n$ and eigenvalues λ_n in the cylinder, the following identities hold under the transformation $n_2 \to -n_2$, $n_3 \to -n_3$

$$\begin{aligned} \mathbf{\Phi}(n_1, -n_2, -n_3) &= \mathbf{\Phi}^*(n_1, n_2, n_3)\,, \\ \lambda(n_1, -n_2, -n_3) &= \lambda(n_1, n_2, n_3)\,. \end{aligned} \tag{3.2}$$

where * designates the complex conjugate (see Section 3, [M-N-B-G] for details). The eigenfunctions $\mathbf{\Phi}(n_1, n_2, n_3)$ involve the radial functions $J_{n_2}(\beta(n_1, n_2, \alpha n_3)r)$ and $J'_{n_2}(\beta(n_1, n_2, \alpha n_3)r)$, with

$$\lambda^2(n_1, n_2, n_3) = \beta^2(n_1, n_2, \alpha n_3) + \alpha^2 n_3^2;$$

$\beta(n_1, n_2, \alpha n_3)$ are discrete, countable roots of equation (3.30) in [M-N-B-G], obtained via an equivalent Sturm-Liouville radial problem. Since the curl eigenfunctions are even in $r \to -r$, $n_1 \to -n_1$, we will extend the indices $n_1 = 1, 2, ..., +\infty$ to $-n_1 = -1, -2, ...$ with the above radial symmetry in mind.

Corollary 3.1 *The 3-wave resonance set K^* is invariant under the symmetries $\sigma_j,\ j = 0, 1, 2, 3$, where*

$$\begin{aligned} \sigma_0(n_1, n_2, n_3) &= (n_1, n_2, n_3), \\ \sigma_1(n_1, n_2, n_3) &= (-n_1, n_2, n_3), \\ \sigma_2(n_1, n_2, n_3) &= (n_1, -n_2, n_3) \\ \sigma_3(n_1, n_2, n_3) &= (n_1, n_2, -n_3)\,. \end{aligned}$$

Remark 3.2 *For $0 < i \leq 3,\ 0 < j \leq 3,\ 0 < l \leq 3$ $\sigma_j^2 = Id,\ \sigma_i \sigma_j = -\sigma_l$ if $i \neq j$ and $\sigma_i \sigma_j \sigma_l = -Id$, for $i \neq j \neq l$. The σ_j do preserve the convolution conditions in K^*.*

We choose an α for which the set K^* is not empty. We further take the hypothesis of a single triple wave resonance (k, m, n), modulo the symmetries σ_j:

Hypothesis 3.3 *K^* is such that there exists a single triple wave number resonance (n, k, m), modulo the symmetries $\sigma_j,\ j = 1, 2, 3$ and $\sigma_j(k) \neq k,\ \sigma_j(m) \neq m,\ \sigma_j(n) \neq n$ for $j = 2$ and $j = 3$.*

Under the above hypothesis, one can demonstrate that the strictly resonant Euler system splits into three uncoupled systems in $\mathbf{C}^3$:

Theorem 3.4 *Under hypothesis 3.3, the resonant Euler system reduces to three uncoupled rigid body systems in* $\mathbf{C}^3$*:*

$$\frac{dU_n}{dt} + i(\lambda_k - \lambda_m)C_{kmn}U_kU_m = 0 \tag{3.3a}$$

$$\frac{dU_k}{dt} - i(\lambda_m - \lambda_n)C_{kmn}U_nU_m^* = 0 \tag{3.3b}$$

$$\frac{dU_m}{dt} - i(\lambda_n - \lambda_k)C_{kmn}U_nU_k^* = 0 \tag{3.3c}$$

where $C_{kmn} = i < \mathbf{\Phi}_k \times \mathbf{\Phi}_m, \mathbf{\Phi}_n^* >$, C_{kmn} *real and the other two uncoupled systems obtained with the symmetries* $\sigma_2(k,m,n)$ *and* $\sigma_3(k,m,n)$*. The energy and the helicity of each subsystem are conserved:*

$$\frac{d}{dt}(U_kU_k^* + U_mU_m^* + U_nU_n^*) = 0,$$

$$\frac{d}{dt}(\lambda_kU_kU_k^* + \lambda_mU_mU_m^* + \lambda_nU_nU_n^*) = 0.$$

Proof. It follows from $U_{-k} = U_k^*$, $\lambda(-k) = \lambda(+k)$, similarly for m and n; and in a very essential way from the antisymmetry of $< \mathbf{\Phi}_k \times \mathbf{\Phi}_m, \mathbf{\Phi}_n^* >$, together with $curl\mathbf{\Phi}_k = \lambda_k\mathbf{\Phi}_k$. That C_{kmn} is real follows from the eigenfunctions explicited in Section 3 of [M-N-B-G]. ■

Remark 3.5 *This deep structure, i.e.* $SO(3;\mathbf{C})$ *rigid body systems in* $\mathbf{C}^3$ *is a direct consequence of the Lamé form of the full 3D Euler equations, cf. Eqs. (1.3) and (2.7), and the nonlinearity* $curl\mathbf{V} \times \mathbf{V}$.

The system (3.3) is equivariant with respect to the symmetry operators

$$(z_1, z_2, z_3) \to (z_1^*, z_2^*, z_3^*),\ (z_1, z_2, z_3) \to (\exp(i\chi_1)z_1, \exp(i\chi_2)z_2, \exp(i\chi_3)z_3)\,,$$

provided $\chi_1 = \chi_2 + \chi_3$. It admits other integrals known as the Manley-Rowe relations (see, for instance [We-Wil]). It differs from the usual 3-wave resonance systems investigated in the literature, such as in [Zak-Man1], [Zak-Man2], [Gu-Ma] in that

(1) helicity is conserved,

(2) dynamics of these resonant systems rigorously "shadow" those of the exact 3D Euler equations, see Theorem 2.10.

Real forms of the system (3.3) are found in Gledzer et al. [G-D-O], corresponding to the exact invariant manifold $U_k \in i\mathbf{R}$, $U_m \in \mathbf{R}$, $U_n \in \mathbf{R}$, albeit without any rigorous asymptotic justification. The $\mathbf{C}^3$ systems (3.3) with *helicity* conservation laws are not discussed in [G-D-O].

The only nontrivial Manley-Rowe conservation laws for the resonant system (3.3), rigid body $SO(3;\mathbf{C})$, which are independent from energy and helicity, are:

$$\frac{d}{dt}\left(r_k r_m r_n \sin(\theta_n - \theta_k - \theta_m)\right) = 0,$$

where $U_j = r_j \exp(i\theta_j),\ j = k, m, n$, and

$$\begin{aligned}\mathcal{E}_1 &= (\lambda_k - \lambda_m) r_n^2 - (\lambda_m - \lambda_n) r_k^2, \\ \mathcal{E}_2 &= (\lambda_m - \lambda_n) r_k^2 - (\lambda_n - \lambda_k) r_m^2.\end{aligned}$$

The resonant system (3.3) is well known to possess hyperbolic equilibria and heteroclinic/homoclinic orbits on the energy surface. We are interested in rigorously proving arbitrary large bursts of enstrophy and higher norms on arbitrarily small time intervals, for properly chosen h/R. To simplify the presentation, we establish the results for the simpler invariant manifold $U_k \in i\mathbf{R}$, and $U_m, U_n \in \mathbf{R}$.

Rescale time as:

$$t \to t/C_{kmn}.$$

Start from the system

$$\begin{aligned}\dot{U}_n + i(\lambda_k - \lambda_m) U_k U_m &= 0 \\ \dot{U}_k - i(\lambda_m - \lambda_n) U_n U_m^* &= 0 \\ \dot{U}_m - i(\lambda_n - \lambda_k) U_n U_k^* &= 0\end{aligned} \tag{3.4}$$

Assume that $U_k \in i\mathbf{R}$ and that $U_m, U_n \in \mathbf{R}$: set $p = iU_k$, $q = U_m$ and $r = U_n$, as well as $\lambda_k = \lambda$, $\lambda_m = \mu$ and $\lambda_n = \nu$: then

$$\begin{aligned}\dot{p} + (\mu - \nu) q r &= 0 \\ \dot{q} + (\nu - \lambda) r p &= 0 \\ \dot{r} + (\lambda - \mu) p q &= 0\end{aligned} \tag{3.5}$$

This system admits two first integrals:

$$\begin{aligned}E &= p^2 + q^2 + r^2 \qquad &\text{(energy)} \\ H &= \lambda p^2 + \mu q^2 + \nu r^2 \qquad &\text{(helicity)}\end{aligned} \tag{3.6}$$

System (3.5) is exactly the $SO(3,\mathbf{R})$ rigid body dynamics Euler equations, with inertia momenta $I_j = \frac{1}{|\lambda_j|},\ j = k, m, n$ [Ar1].

Lemma 3.6 ([Ar1], [G-D-O]) *With the ordering $\lambda_k > \lambda_m > \lambda_n$, i.e. $\lambda > \mu > \nu$, the equilibria $(0, \pm 1, 0)$ are hyperbolic saddles on the unit energy sphere, and the equilibria $(\pm 1, 0, 0)$, $(0, 0, \pm 1)$ are centers. There exist equivariant families of heteroclinic connections between $(0, +1, 0)$ and $(0, -1, 0)$. Each pair of such connections correspond to equivariant homoclinic cycles at $(0, 1, 0)$ and $(0, -1, 0)$.*

We investigate bursting dynamics along orbits with large periods, with initial conditions close to the hyperbolic point $(0, E(0), 0)$ on the energy sphere E. We choose resonant triads such that $\lambda_k > 0,\ \lambda_n < 0, \lambda_k \sim |\lambda_n|,\ |\lambda_m| \ll \lambda_k$, equivalently:

$$\lambda > \mu > \nu,\ \lambda\nu < 0,\ |\mu| \ll \lambda \text{ and } \lambda \sim |\nu|. \tag{3.7}$$

Lemma 3.7 *There exist h/R with $K^* \neq \varnothing$, such that*

$$\lambda_k > \lambda_m > \lambda_n,\ \lambda_k\lambda_n < 0,\ |\lambda_m| \ll \lambda_k \textit{ and } \lambda_k \sim |\lambda_n|.$$

Remark 3.8 *Together with the polarity $\pm$ of the curl eigenvalues, these are 3-wave resonances where two of the eigenvalues are much larger in moduli than the third one. In the limit $|k|,\ |m|,\ |n| \gg 1, \lambda_k \sim \pm|k|,\ \lambda_m \sim \pm|m|,\ \lambda_n \sim \pm|n|$, the eigenfunctions $\mathbf{\Phi}$ have leading asymptotic terms which involve cosines and sines periodic in r, cf. Section 3* [M-N-B-G]. *In the strictly resonant equations (2.30), the summation over the quadratic terms becomes an asymptotic convolution in $n_1 = k_1 + n_1$. The resonant three waves in Lemma 3.7 are equivalent to Fourier triads $k + m = n$, with $|k| \sim |n|$ and $|m| \ll |k|,\ |n|$, in periodic lattices. In the physics of spectral theory of turbulence* [Fri], [Les]*, these are exactly the triads responsible from transfer of energy between large scales and small scales. These are the triads which have hampered mathematical efforts at proving the global regularity of the Cauchy problem for 3D Navier-Stokes equations in periodic lattices* [Fe].

Proof of Lemma 3.7 ([M-N-B-G]) The transcendental dispersion law for 3-waves in K^* for cylindrical domains, is a polynomial of degree four in $\vartheta_3 = 1/h^2$:

$$\tilde{P}(\vartheta_3) = \tilde{P}_4\vartheta_3^4 + \tilde{P}_3\vartheta_3^3 + \tilde{P}_2\vartheta_3^2 + \tilde{P}_1\vartheta_3 + \tilde{P}_0 = 0, \tag{3.8}$$

with $n_2 = k_2 + m_2$ and $n_3 = k_3 + m_3$.

Then with $h_k = \frac{\beta^2(k_1,k_2,\alpha k_3)}{k_3^2},\ h_m = \frac{\beta^2(m_1,m_2,\alpha m_3)}{m_3^2},\ h_n = \frac{\beta^2(n_1,n_2,\alpha n_3)}{n_3^2}$, cf. the radial Sturm-Liouville problem in Section 3, [M-N-B-G], the coefficients of $\tilde{P}(\vartheta_3)$ are given by:

$$\begin{aligned}
\tilde{P}_4 &= -3,\\
\tilde{P}_3 &= -4(h_k + h_m + h_n),\\
\tilde{P}_2 &= -6(h_k h_m + h_k h_n + h_m h_n),\\
\tilde{P}_1 &= -12 h_k h_m h_n,\\
\tilde{P}_0 &= h_m^2 h_n^2 + h_k^2 h_n^2 + h_m^2 h_k^2 - 2(h_k h_m h_n^2 + h_k h_n h_m^2 + h_m h_n h_k^2).
\end{aligned}$$

Similar formulas for the periodic lattice domain were first derived in [B-M-N2], [B-M-N3], [B-M-N4]. In cylindrical domains the resonance condition for K^* is identical to

$$\pm\frac{1}{\sqrt{\vartheta_3 + h_k}} \pm \frac{1}{\sqrt{\vartheta_3 + h_m}} \pm \frac{1}{\sqrt{\vartheta_3 + h_n}} = 0,$$

with $\vartheta_3 = \frac{1}{h^2},\ h_k = \beta^2(k)/k_3^2,\ h_m = \beta^2(m)/m_3^2,\ h_n = \beta^2(n)/n_3^2$; Eq. (3.8) is the equivalent rational form.

From the asymptotic formula (3.44) in [M-N-B-G], for large β:

$$\beta(n_1, n_2, n_3) \sim n_1\pi + n_2\frac{\pi}{2} + \frac{\pi}{4} + \psi, \tag{3.9}$$

where $\psi = 0$ if $\lim \frac{m_2}{m_3}\frac{h}{2\pi} = 0$ (e.g. h fixed, $m_2/m_3 \to 0$) and $\psi = \pm\frac{\pi}{2}$ if $\lim \frac{m_2}{m_3}\frac{h}{2\pi} = \pm\infty$ (e.g. $\frac{m_2}{m_3}$ fixed, $h \to \infty$). The proof is completed by taking leading terms $\tilde{P}_0 + \vartheta_3\tilde{P}_1$ in (3.8), $\vartheta_3 = \frac{1}{h^2} \ll 1$, and $m_2 = 0,\ k_2 = \mathcal{O}(1),\ n_2 = \mathcal{O}(1)$. ∎

We now state a theorem for bursting of the $\mathbf{H}^3$ norm in arbitrarily small times, for initial data close to the hyperbolic point $(0, E(0), 0)$:

Theorem 3.9 *(Bursting dynamics in $\mathbf{H}^3$). Let $\lambda > \mu > \nu,\ \lambda\nu < 0,\ |\mu| \ll \lambda$ and $\lambda \sim |\nu|$. Let $W(t) = \lambda^6 p(t)^2 + \mu^6 q(t)^2 + \nu^6 r(t)^2$ the $\mathbf{H}^3$-norm squared of an orbit of (3.5). Choose initial data such that: $W(0) = \lambda^6 p(0)^2 + \mu^6 q(0)^2$ with $\lambda^6 p(0)^2 \sim \frac{1}{2}W(0)$ and $\mu^6 q(0)^2 \sim \frac{1}{2}W(0)$. Then there exists $t^* > 0$, such that*

$$W(t) \geq \frac{1}{4}\left(\frac{\lambda}{\mu}\right)^6 W(0)$$

where $t^* \leq \frac{6}{\sqrt{W(0)}}\mu^2 Ln(\lambda/|\mu|)(\lambda/|\mu|)^{-1}$.

Remark 3.10 *Under the conditions of Lemma 3.7, $\left(\frac{\lambda}{\mu}\right)^6 \gg 1$, whereas $\mu^2(Ln(\lambda/|\mu|))(\lambda/|\mu|)^{-1} \ll 1$. Therefore, over a small time interval of length $O(\mu^2(Ln(\lambda/|\mu|))(\lambda/|\mu|)^{-1}) \ll 1$, the ratio $||\mathbf{U}(t)||_{H^3}/||\mathbf{U}(0)||_{H^3}$ grows up to a maximal value $O\left((\lambda/|\mu|)^3\right) \gg 1$. Since the orbit is periodic, the $\mathbf{H}^3$ semi-norm eventually relaxes to its initial state after some time (this being a manifestation of the time-reversibility of the Euler flow on the energy sphere). The "shadowing" theorem 2.10 with $s > 7/2$ ensures that the full, original 3D Euler dynamics, with the same initial conditions, will undergo the same type of burst. Notice that, with the definition (1.13) of $\|\cdot\|_{H^s}$, one has*

$$||\Omega\mathbf{e}_3 \times y||_{H^3} = ||curl^3(\Omega\mathbf{e}_3 \times y)||_{L^2} = 0\,.$$

Hence the solid rotation part of the original 3D Euler solution does not contribute to the ratio $||\mathbf{V}(t)||_{H^3}/||\mathbf{V}(0)||_{H^3}$.

Theorem 3.11 *(Bursting dynamics of the enstrophy). Under the same conditions for the 3-wave resonance, let $\Xi(t) = \lambda^2 p(t)^2 + \mu^2 q(t)^2 + \nu^2 r(t)^2$ the enstrophy. Choose initial data such that $\Xi(0) = \lambda^2 p(0)^2 + \mu^2 q(0)^2 + \nu^2 r(0)^2$ with $\lambda^2 p(0)^2 \sim \frac{1}{2}\Xi(0),\ \mu^2 q(0)^2 \sim \frac{1}{2}\Xi(0)$. Then there exists $t^{**} > 0$, such that*

$$\Xi(t^{**}) \geq \left(\frac{\lambda}{\mu}\right)^2$$

where $t^{**} \leq \frac{1}{\sqrt{2}}\frac{1}{\sqrt{\Xi(0)}}Ln\left(\lambda/|\mu|\right)\left(\lambda/|\mu|\right)^{-1}$.

Remark 3.12 *It is interesting to compare this mechanism for bursts with earlier results in the same direction obtained by DiPerna and Lions. Indeed, for each $p \in (1, \infty)$, each $\delta \in (0, 1)$ and each $t > 0$, Di Perna and Lions* [DiPe-Li] *constructed examples of 2D-3 components solutions to Euler equations such that*

$$||\mathbf{V}(0)||_{W^{1,p}} \leq \epsilon \quad \textit{while} \quad ||\mathbf{V}(t)||_{W^{1,p}} \geq 1/\delta\,.$$

Their examples essentially correspond to shear flows of the form

$$\mathbf{V}(t, x_1, x_2) = \begin{pmatrix} u(x_2) \\ 0 \\ w(x_1 - tu(x_2), x_2) \end{pmatrix}$$

where $u \in W^{1,p}_{x_2}$ while $w \in W^{1,p}_{x_2}$. Obviously

$$curl\mathbf{V}(t, x_1, x_2) = \begin{pmatrix} (\partial_2 - tu'(x_2)\partial_1)w(x_1 - tu(x_2), x_2) \\ -\partial_1 w(x_1 - tu(x_2), x_2) \\ -u'(x_2) \end{pmatrix}$$

Thus, all components in $curl\mathbf{V}(t, x_1, x_2)$ belong to L^p_{loc}, except for the term

$$-tu'(x_2)\partial_1 w(x_1 - tu(x_2), x_2)\,.$$

For each $t > 0$, this term belongs to L^p for all choices of the functions $u \in W^{1,p}_{x_2}$ and $w \in W^{1,p}_{x_1,x_2}$ if and only if $p = \infty$. Whenever $p < \infty$, DiPerna and Lions construct their examples as some smooth approximation of the situation above in the strong $W^{1,p}$ topology.

In other words, the DiPerna-Lions construction works only in cases where the initial vorticity does not belong to an algebra — specifically to L^p, which is not an algebra unless $p = \infty$.

The type of burst obtained in our construction above is different: in that case, the original vorticity belongs to the Sobolev space H^2, which is an algebra in space dimension 3*. Similar phenomena are observed in all Sobolev spaces H^β with $\beta \geq 2$ — which are also algebras in space dimension 3.*

In other words, our results complement those of DiPerna-Lions on bursts in higher order Sobolev spaces, however at the expense of using more intricate dynamics.

We proceed to the proofs of Theorem 3.9 and 3.11. We are interested in the evolution of

$$\Xi = \lambda^2 p^2 + \mu^2 q^2 + \nu^2 r^2 \qquad \text{(enstrophy)} \tag{3.10}$$

Compute

$$\dot{\Xi} = -2\left(\lambda^2(\mu - \nu) + \mu^2(\nu - \lambda) + \nu^2(\lambda - \mu)\right) pqr \tag{3.11}$$

then

$$\dot{(pqr)} = -(\mu - \nu)q^2 r^2 - (\nu - \lambda)r^2 p^2 - (\lambda - \mu)p^2 q^2 \tag{3.12}$$

Using the first integrals above, one has

$$Van \begin{pmatrix} p^2 \\ q^2 \\ r^2 \end{pmatrix} = \begin{pmatrix} E \\ H \\ \Xi \end{pmatrix} \tag{3.13}$$

where Van is the Vandermonde matrix

$$Van = \begin{pmatrix} 1 & 1 & 1 \\ \lambda & \mu & \nu \\ \lambda^2 & \mu^2 & \nu^2 \end{pmatrix}$$

For $\lambda \neq \mu \neq \nu \neq \lambda$, this matrix is invertible and

$$Van^{-1} = \begin{pmatrix} \frac{\mu\nu}{(\lambda-\mu)(\lambda-\nu)} & \frac{-(\mu+\nu)}{(\lambda-\mu)(\lambda-\nu)} & \frac{1}{(\lambda-\mu)(\lambda-\nu)} \\ \frac{\nu\lambda}{(\mu-\nu)(\mu-\lambda)} & \frac{-(\nu+\lambda)}{(\mu-\nu)(\mu-\lambda)} & \frac{1}{(\mu-\nu)(\mu-\lambda)} \\ \frac{\lambda\mu}{(\nu-\lambda)(\nu-\mu)} & \frac{-(\lambda+\mu)}{(\nu-\lambda)(\nu-\mu)} & \frac{1}{(\nu-\lambda)(\nu-\mu)} \end{pmatrix}$$

Hence

$$\begin{aligned} p^2 &= \frac{1}{(\lambda-\mu)(\lambda-\nu)} \left(\Xi - (\mu+\nu)H + \mu\nu E\right) \\ q^2 &= \frac{1}{(\mu-\nu)(\mu-\lambda)} \left(\Xi - (\nu+\lambda)H + \nu\lambda E\right) \\ r^2 &= \frac{1}{(\nu-\lambda)(\nu-\mu)} \left(\Xi - (\lambda+\mu)H + \lambda\mu E\right) \end{aligned} \tag{3.14}$$

so that

$$\begin{aligned} (\mu-\nu)q^2r^2 &= -\frac{\left(\Xi - (\nu+\lambda)H + \nu\lambda E\right)\left(\Xi - (\lambda+\mu)H + \lambda\mu E\right)}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)} \\ (\nu-\lambda)r^2p^2 &= -\frac{\left(\Xi - (\lambda+\mu)H + \lambda\mu E\right)\left(\Xi - (\mu+\nu)H + \mu\nu E\right)}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)} \\ (\lambda-\mu)p^2q^2 &= -\frac{\left(\Xi - (\mu+\nu)H + \mu\nu E\right)\left(\Xi - (\nu+\lambda)H + \nu\lambda E\right)}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)} \end{aligned}$$

Later on, we shall use the notations

$$\begin{aligned} x_-(\lambda,\mu,\nu) &= (\mu+\nu)H - \mu\nu E \\ x_0\,(\lambda,\mu,\nu) &= (\mu+\lambda)H - \mu\lambda E \\ x_+(\lambda,\mu,\nu) &= (\lambda+\nu)H - \lambda\nu E \end{aligned} \tag{3.15}$$

Therefore, we find that Ξ satisfies the second order ODE

$$\begin{aligned} \ddot{\Xi} = &- 2K_{\lambda,\mu,\nu}\left((\Xi - x_-(\lambda,\mu,\nu))(\Xi - x_0(\lambda,\mu,\nu))\right. \\ &\left.+(\Xi - x_0(\lambda,\mu,\nu))(\Xi - x_+(\lambda,\mu,\nu)) + (\Xi - x_+(\lambda,\mu,\nu))(\Xi - x_0(\lambda,\mu,\nu))\right) \end{aligned}$$

which can be put in the form

$$\ddot{\Xi} = -2K_{\lambda,\mu,\nu}P'_{\lambda,\mu,\nu}(\Xi) \tag{3.16}$$

where $P_{\lambda,\mu,\nu}$ is the cubic

$$P_{\lambda,\mu,\nu}(X) = (X - x_-(\lambda,\mu,\nu))(X - x_0(\lambda,\mu,\nu))(X - x_+(\lambda,\mu,\nu)) \tag{3.17}$$

and

$$K_{\lambda,\mu,\nu} = \frac{\lambda^2(\mu-\nu) + \mu^2(\nu-\lambda) + \nu^2(\lambda-\mu)}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)} \tag{3.18}$$

In the sequel, we assume that the initial data for (p, q, r) is such that

$$r(0) = 0\,, \qquad p(0)(q(0) \neq 0$$

Let us compute

$$\begin{aligned} x_-(\lambda,\mu,\nu) &= \lambda\nu p(0)^2 + \mu^2 q(0)^2 + \mu(\lambda-\nu)p(0)^2 \\ x_0\,(\lambda,\mu,\nu) &= \lambda^2 p(0)^2 + \mu^2 q(0)^2 \\ x_+(\lambda,\mu,\nu) &= \lambda^2 p(0)^2 + \left(\frac{\nu+\lambda}{\mu} - \frac{\nu\lambda}{\mu^2}\right)\mu^2 q(0)^2 \end{aligned} \tag{3.19}$$

We shall also assume that

$$\lambda > \mu > \nu\,, \qquad \lambda\nu < 0\,, \qquad |\mu| \ll \lambda \text{ and } \lambda \sim |\nu| \tag{3.20}$$

Then $K_{\lambda,\mu,\nu} > 0$ — in fact $K_{\lambda,\mu,\nu} \sim 2$, and Ξ is a periodic function of t such that

$$\inf_{t\in\mathbf{R}} \Xi(t) = x_0(\lambda,\mu,\nu)\,, \quad \sup_{t\in\mathbf{R}} \Xi(t) = x_+(\lambda,\mu,\nu) \tag{3.21}$$

with half-period

$$T_{\lambda,\mu,\nu} = \frac{1}{2\sqrt{K_{\lambda,\mu,\nu}}} \int_{x_0(\lambda,\mu,\nu)}^{x_+(\lambda,\mu,\nu)} \frac{dx}{\sqrt{-P_{\lambda,\mu,\nu}(x)}} \tag{3.22}$$

We are interested in the growth of the (squared) $\mathbf{H}^3$ norm

$$W(t) = \lambda^6 p(t)^2 + \mu^6 q(t)^2 + \nu^6 r(t)^2 \tag{3.23}$$

Expressing p^2, q^2 and r^2 in terms of E, H and Ξ, it is found that

$$W = \frac{\lambda^6(\Xi - x_-(\lambda,\mu,\nu))}{(\lambda-\mu)(\lambda-\nu)} + \frac{\mu^6(\Xi - x_+(\lambda,\mu,\nu))}{(\mu-\nu)(\mu-\lambda)} + \frac{\nu^6(\Xi - x_0(\lambda,\mu,\nu))}{(\nu-\lambda)(\nu-\mu)} \tag{3.24}$$

Hence, when $\Xi = x_+(\lambda,\mu,\nu)$, then

$$\begin{aligned} W &= \frac{\lambda^6(x_+(\lambda,\mu\,\nu) - x_-(\lambda,\mu,\nu))}{(\lambda-\mu)(\lambda-\nu)} + \frac{\nu^6(x_+(\lambda,\mu\,\nu) - x_0(\lambda,\mu,\nu))}{(\nu-\lambda)(\nu-\mu)} \\ &\geq \frac{\lambda^6(x_+(\lambda,\mu\,\nu) - x_-(\lambda,\mu,\nu))}{(\lambda-\mu)(\lambda-\nu)} \end{aligned}$$

Let us compute

$$x_+(\lambda,\mu\,\nu) - x_-(\lambda,\mu,\nu) = (\lambda-\mu)(\lambda-\nu)p(0)^2 + \left(\frac{\nu+\lambda}{\mu} - \frac{\nu\lambda}{\mu} - 1\right)\mu^2 q(0)^2$$
$$\gtrsim -\nu\lambda q(0)^2 \sim \lambda^2 q(0)^2 \tag{3.25}$$

We shall pick the initial data such that

$$W(0) = \lambda^6 p(0)^6 + \mu^6 q(0)^6 \text{ with } \lambda^6 p(0)^2 \sim \tfrac{1}{2}W(0) \text{ and } \mu^6 q(0)^2 \sim \tfrac{1}{2}W(0) \tag{3.26}$$

Hence, when Ξ reaches $x_+(\lambda,\mu,\nu)$, one has

$$W \gtrsim \frac{\lambda^8 q(0)^2}{(\lambda-\mu)(\lambda-\nu)} \sim \tfrac{1}{2}\frac{\lambda^8}{\mu^6(\lambda-\mu)(\lambda-\nu)}W(0) \sim \tfrac{1}{4}\frac{\lambda^6}{\mu^6}W(0)\,. \tag{3.27}$$

Hence W jumps from $W(0)$ to a quantity $\sim \frac{1}{4}\frac{\lambda^6}{\mu^6}W(0)$ in an interval of time that does not exceed one period of the Ξ motion, i.e. $2T_{\lambda,\mu,\nu}$. Let us estimate this interval of time. We recall the asymptotic equivalent for the period of an elliptic integral in the modulus 1 limit.

Lemma 3.13 *Assume that $x_- < x_0 < x_+$. Then*

$$\int_{x_0}^{x_+} \frac{dx}{\sqrt{(x-x_-)(x-x_0)(x_+-x)}} \sim \frac{1}{\sqrt{x_+-x_-}} \ln\left(\frac{1}{1-\sqrt{\frac{x_+-x_0}{x_+-x_-}}}\right)$$

uniformly in x_-, x_0, and x_+ as $\frac{x_+-x_0}{x_+-x_-} \to 1$.

Here

$$\frac{1}{\sqrt{x_+(\lambda,\mu,\nu)-x_-(\lambda,\mu,\nu)}} \lesssim \frac{1}{\sqrt{\lambda^2 q(0)^2}} \sim \frac{|\mu|^3}{\lambda}\sqrt{\frac{2}{W(0)}}$$

Next

$$x_0(\lambda,\mu,\nu) - x_-(\lambda,\mu,\nu) = (\lambda-\mu)(\lambda-\nu)p(0)^2 \tag{3.28}$$

so that

$$\frac{1}{1-\sqrt{\frac{x_+-x_0}{x_+-x_-}}} \sim \frac{1}{1-\sqrt{1-\frac{(\lambda-\mu)(\lambda-\nu)p(0)^2}{(\lambda-\mu)(\lambda-\nu)p(0)^2+(\mu(\nu+\lambda)-\nu\lambda-\mu^2)q(0)^2}}}$$
$$\sim \frac{(\lambda-\mu)(\lambda-\nu)p(0)^2+(\mu(\nu+\lambda)-\nu\lambda-\mu^2)q(0)^2}{2(\lambda-\mu)(\lambda-\nu)p(0)^2}$$
$$\sim \frac{q(0)^2}{2p(0)^2} \sim \frac{1}{2}\frac{W(0)/2\mu^6}{W(0)/2\lambda^6} = \frac{\lambda^6}{2\mu^6}$$

Hence

$$2T_{\lambda,\mu,\nu} \lesssim \frac{2}{\sqrt{W(0)}} \frac{\mu^3}{\lambda} \ln\left(\frac{\lambda^6}{2\mu^6}\right) \leq \frac{12}{\sqrt{W(0)}} \frac{|\mu|^3}{\lambda} \ln\left(\frac{\lambda}{\mu}\right) \tag{3.29}$$

Conclusion: collecting (3.26), (3.27) and (3.29), we see that the squared $\mathbf{H}^3$ norm W varies from $W(0)$ to a quantity $\sim \rho^6 W(0)$ in an interval of time $\lesssim \frac{12}{\sqrt{W(0)}} \frac{\mu^2 \ln \rho}{\rho}$. (Here $\rho = \lambda/\mu$).

We now proceed to obtain similar bursting estimates for the enstrophy. We return to (3.21) and (3.22). Pick the initial data so that

$$\Xi(0) = \lambda^2 p(0)^2 + \mu^2 q(0)^2 \text{ with } \lambda^2 p(0)^2 \sim \frac{1}{2}\Xi(0) \text{ and } \mu^2 q(0)^2 \sim \frac{1}{2}\Xi(0).$$

Then

$$\begin{aligned} & x_+(\lambda,\mu,\nu) - x_-(\lambda,\mu,\nu) \\ & \qquad = (\lambda-\mu)(\lambda-\nu)p(0)^2 + \left(\frac{\nu+\lambda}{\mu} - \frac{\nu\lambda}{\mu^2} - 1\right)\mu^2 q(0)^2 \\ & \qquad \sim 2\lambda^2 p(0)^2 + \lambda^2 q(0)^2 \sim \frac{\lambda^2}{\mu^2}\Xi(0) \end{aligned}$$

while

$$x_0(\lambda,\mu,\nu) - x_-(\lambda,\mu,\nu) = (\lambda-\mu)(\lambda-\nu)p(0)^2 \sim 2\lambda^2 p(0)^2 \sim \Xi(0).$$

Hence, in the limit as $\rho = \lambda/|\mu| \to +\infty$, one has

$$\begin{aligned} 2T_{\lambda,\mu,\nu} &\sim \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{\rho^2 \Xi(0)}} \ln \frac{1}{1 - \sqrt{1 - \frac{\Xi(0)}{\frac{1}{2}\rho^2 \Xi(0)}}} \\ &= \frac{1}{2\sqrt{2\Xi(0)}} \frac{1}{\rho} \ln \frac{1}{1-\sqrt{1-2\rho^{-2}}} \sim \frac{1}{\sqrt{2\Xi(0)}} \frac{\ln \rho}{\rho}. \end{aligned}$$

And Ξ varies from

$$x_0(\lambda,\mu,\nu) = \Xi(0) \text{ to } x_+(\lambda,\mu,\nu) \sim \rho^2 \Xi(0)$$

on an interval of time of length $T_{\lambda,\mu,\nu}$. ■

4 Strictly resonant Euler systems: the case of 3-waves resonances on small-scales

4.1 Infinite dimensional uncoupled $SO(3)$ systems

In this section, we consider the 3-wave resonant set K^* when

$$|k|^2, \ |m|^2, \ |n|^2 \geq \frac{1}{\eta^2}, \ 0 < \eta \ll 1,$$

i.e. 3-wave resonances on small scales; here $|k|^2 = k_1^2 + k_2^2 + k_3^2$, where (k_1, k_2, k_3) index the curl eigenvalues, and similarly for $|m|^2,\ |n|^2$. Recall that $k_2 + m_2 = n_2,\ k_3 + m_3 = n_3$ (exact convolutions), but that the summation on $k_1,\ m_1$ on the right hand side of Eqs. (2.30) is not a convolution. However, for $|k|^2,\ |m|^2,\ |n|^2 \geq \frac{1}{\eta^2}$, the summation in $k_1,\ m_1$ becomes an asymptotic convolution. First:

Proposition 4.1 *The set K^* restricted to $|k|^2, |m|^2, |n|^2 \geq \frac{1}{\eta^2}, \forall \eta, 0 < \eta \ll 1$ is not empty: there exist at least one h/R with resonant three waves satisfying the above small scales condition.*

Proof. We follow the algebra of the exact transcendental dispersion law (3.8) derived in the proof of Lemma 3.7. Note that $\tilde{P}(\vartheta_3) < 0$ for $\vartheta_3 = \frac{1}{h^2}$ large enough. We can choose $h_m = \frac{\beta^2(m_1, m_2, \alpha m_3)}{m_3^2} = 0$, say in the specific limit $\frac{h}{2\pi m_3} \to 0$, and $\beta(m_1, m_2, \alpha m_3) \sim m_1 \pi + m_2 \frac{\pi}{2} + \frac{\pi}{4}$. Then $\tilde{P}_0 = h_k^2 h_n^2 > 0$ and $\tilde{P}(\vartheta_3)$ must possess at least one (transcendental) root $\vartheta_3 = \frac{1}{h^2}$. ■

In the above context, the radial components of the curl eigenfunctions involve cosines and sines in $\frac{\beta r}{R}$ (cf. Section 3, [M-N-B-G]) and the summation in $k_1,\ m_1$ on the right hand side of the resonant Euler equations (2.30) becomes an *asymptotic convolution*. The rigorous asymptotic convolution estimates are highly technical and detailed in [Fro-M-N]. The 3-wave resonant systems for $|k|^2,\ |m|^2,\ |n|^2 \geq \frac{1}{\eta^2}$ are equivalent to those of an equivalent periodic lattice $[0, 2\pi] \times [0, 2\pi] \times [0, 2\pi h],\ \vartheta_3 = \frac{1}{h^2}$; the resonant three wave relation becomes:

$$\pm \left(\vartheta_3 + \vartheta_1 \frac{n_1^2}{n_3^2} + \vartheta_2 \frac{n_2^2}{n_3^2}\right)^{-\frac{1}{2}} \pm \left(\vartheta_3 + \vartheta_1 \frac{k_1^2}{k_3^2} + \vartheta_2 \frac{k_2^2}{k_3^2}\right)^{-\frac{1}{2}} \pm \left(\vartheta_3 + \vartheta_1 \frac{m_1^2}{m_3^2} + \vartheta_2 \frac{m_2^2}{m_3^2}\right)^{-\frac{1}{2}} = 0, \tag{4.1a}$$

$$k + m = n,\ k_3 m_3 n_3 \neq 0. \tag{4.1b}$$

The algebraic geometry of these rational 3-wave resonance equations has been investigated in depth in [B-M-N3] and [B-M-N4]. Here $\vartheta_1,\ \vartheta_2,\ \vartheta_3$ are periodic lattice parameters; in the small-scales cylindrical case, $\vartheta_1 = \vartheta_2 = 1$ (after rescaling of $n_2,\ k_2,\ m_2$), $\vartheta_3 = 1/h^2$, h height. Based on the algebraic geometry of "resonance curves" in [B-M-N3], [B-M-N4], we investigate the resonant 3D Euler equations (2.30) in the equivalent *periodic lattices*.

First, triplets (k, m, n) solution of (4.1) are invariant under the reflection symmetries $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ defined in Corollary 3.1 and Remark 3.2: $\sigma_0 = Id$, $\sigma_j(k) = (\epsilon_{i,j} k_i),\ 1 \leq i \leq 3,\ \epsilon_{i,j} = +1$ if $i \neq j$, $\epsilon_{i,j} = -1$ if $i = j$, $1 \leq j \leq 3$. Second the set K^* in (4.1) is invariant under the homothetic transformations:

$$(k, m, n) \to (\gamma k, \gamma m, \gamma n),\ \gamma \text{ rational}. \tag{4.2}$$

The resonant triplets lie on projective lines in the wavenumber space, with equivariance under $\sigma_j,\ 0 \le j \le 3$ and γ-rescaling. For every given equivariant family of such projective lines, the resonant curve is the graph of $\frac{\vartheta_3}{\vartheta_1}$ versus $\frac{\vartheta_2}{\vartheta_1}$, for parametric domain resonances in $\vartheta_1,\ \vartheta_2,\ \vartheta_3$.

Lemma 4.2 *(p.17,* [B-M-N4]*). For every equivariant (k, m, n), the resonant curve in the quadrant $\vartheta_1 > 0, \vartheta_2 > 0, \vartheta_3 > 0$ is the graph of a smooth function $\vartheta_3/\vartheta_1 \equiv F(\vartheta_2/\vartheta_1)$ intersected with the quadrant.*

Theorem 4.3 *(p.19,* [B-M-N4]*). A resonant curve in the quadrant $\vartheta_3/\vartheta_1 > 0$, $\vartheta_2/\vartheta_1 > 0$ is called irreducible if:*

$$det \begin{pmatrix} k_3^2 & k_2^2 & k_1^2 \\ m_3^2 & m_2^2 & m_1^2 \\ n_3^2 & n_2^2 & n_1^2 \end{pmatrix} \neq 0. \tag{4.3}$$

An irreducible resonant curve is uniquely characterized by six non-negative algebraic invariants $\mathcal{P}_1,\ \mathcal{P}_2,\ \mathcal{R}_1,\ \mathcal{R}_2,\ \mathcal{S}_1,\ \mathcal{S}_2$, such that

$$\left\{\frac{n_1^2}{n_3^3}, \frac{n_2^2}{n_3^2}\right\} = \left\{\mathcal{P}_1^2, \mathcal{P}_2^2\right\}, \quad \left\{\frac{k_1^2}{k_3^3}, \frac{k_2^2}{k_3^2}\right\} = \left\{\mathcal{R}_1^2, \mathcal{R}_2^2\right\}, \quad \left\{\frac{m_1^2}{m_3^3}, \frac{m_2^2}{m_3^2}\right\} = \left\{\mathcal{S}_1^2, \mathcal{S}_2^2\right\},$$

and permutations thereof.

Lemma 4.4 *(p. 25,* [B-M-N4]*). For resonant triplets (k, m, n) associated to a given irreducible resonant curve, that is verifying Eq. (4.3), consider the convolution equation $n = k + m$. Let $\sigma_i(n) \neq n,\ \forall i,\ 1 \le i \le 3$. Then there are no more that two solutions (k, m) and (m, k), for a given n, provided the six non-degeneracy conditions (3.39)-(3.44) in* [B-M-N4] *for the algebraic invariants of the irreducible curve are verified.*

For more details on the technical non-degeneracy conditions, see the Appendix. An exhaustive algebraic geometric investigation of all solutions to $n = k + m$ on irreducible resonant curves is found in [B-M-N4]. The essence of the above lemma lies in that given such an irreducible, "non-degenerate" triplet (k, m, n) on K^*, all other triplets on the same irreducible resonant curves are exhaustively given by the equivariant projective lines:

$$(k, m, n) \quad \to \quad (\gamma k, \gamma m, \gamma n), \quad \text{for some } \gamma \text{ rational}, \tag{4.4}$$

$$(k, m, n) \quad \to \quad (\sigma_j k, \sigma_j m, \sigma_j n), \quad j = 1, 2, 3, \tag{4.5}$$

and permutations of k and m in the above. Of course the homothety γ and the σ_j symmetries preserve the convolution. This context of irreducible, "non-degenerate" resonant curves yields an infinite dimensional, uncoupled system of rigid body $SO(3; \mathbf{R})$ and $SO(3; \mathbf{C})$ dynamics for the 3D resonant Euler equations (2.30).

Theorem 4.5 *For any irreducible triplet (k,m,n) which satisfy Theorem 4.3, and under the "non-degeneracy" conditions of Lemma 4.4 (cf. Appendix), the resonant Euler equations split into the infinite, countable sequence of uncoupled $SO(3;\mathbf{R})$ systems:*

$$\dot{a}_k = \Gamma_{kmn}(\lambda_m - \lambda_n)a_m a_n, \tag{4.6a}$$

$$\dot{a}_m = \Gamma_{kmn}(\lambda_n - \lambda_k)a_n a_k, \tag{4.6b}$$

$$\dot{a}_n = \Gamma_{kmn}(\lambda_k - \lambda_m)a_k a_m, \tag{4.6c}$$

for all $(k,m,n) = \gamma(\sigma_j(k^*), \sigma_j(m^*), \sigma_j(n^*)),\ \gamma = \pm1, \pm2, \pm3...,\ 0 \le j \le 3.$ (4.7)

k^*, m^*, n^* are some relatively prime integer vectors in $\mathbf{Z}^3$ characterizing the equivariant family of projective lines (k,m,n); $\Gamma_{kmn} = i < \mathbf{\Phi}_k \times \mathbf{\Phi}_m, \mathbf{\Phi}_n^* >$, Γ_{kmn} real.

Proof. Theorem 4.5 is a simpler version for invariant manifolds of more general $SO(3;\mathbf{C})$ systems. It is a straightforward corollary of Proposition 3.2, Proposition 3.3, Theorem 3.3, Theorem 3.4 and Theorem 3.5 in [B-M-N4]. The latter article did not explicit the resonant equations and did not use the curl-helicity algebra fundamentally underlying this present work. Rigorously asymptotic infinite countable sequences of uncoupled $SO(3;\mathbf{R})$, $SO(3;\mathbf{C})$ systems are not derived via the usual harmonic analysis tools of Fourier modes, in the 3D Euler context. Polarization of curl eigenvalues and eigenfunctions and helicity play an essential role.

Corollary 4.6 *Under the conditions $\lambda_{n^*} - \lambda_{k^*} > 0,\ \lambda_{k^*} - \lambda_{m^*} > 0$, the resonant Euler systems (4.6) admit a disjoint, countable family of homoclinic cycles. Moreover, under the conditions $\lambda_{n^*} \gg +1,\ \lambda_{m^*} \ll -1,\ |\lambda_{k^*}| \ll \lambda_{n^*}$, each subsystem (4.6) possesses orbits whose $\mathbf{H}^s$ norms, $s \ge 1$, burst arbitrarily large in arbitrarily small times.*

Remark 4.7 *One can prove that there exists some $\Gamma_{max},\ 0 < \Gamma_{max} < \infty$, such that $|\Gamma_{kmn}| < \Gamma_{max}$, for all (k,m,n) on the equivariant projective lines defined by (4.7). Systems (4.6) "freeze" cascades of energy; their total enstrophy $\Xi(t) = \sum_{(k,m,n)}(\lambda_k^2 a_k^2(t) + \lambda_m^2 a_m^2(t) + \lambda_n^2 a_n^2(t))$ remains bounded, albeit with large bursts of $\Xi(t)/\Xi(0)$, on the reversible orbits topologically close to the homoclinic cycles.*

4.2 Coupled $SO(3)$ rigid body resonant systems

We now derive a new resonant Euler system which couples **two** $SO(3;\mathbf{R})$ rigid bodies via a common principle axis of inertia and a common moment of inertia. This 5-dimensional system conserves energy, helicity, and is rather interesting in that dynamics on its homoclinic manifolds show bursting cascades of enstrophy to the smallest scale in the resonant set. We consider the equivalent periodic lattice geometry under the conditions of Proposition 4.1.

In Appendix, we prove that for an "irreducible" 3-wave resonant set which now satisfies the algebraic "degeneracy" (A-4), there exist exactly two "primitive" resonant triplets (k, m, n) and $(\tilde{k}, \tilde{m}, n)$, where $k,\ m,\ \tilde{k},\ \tilde{m}$ are relative prime integer valued vectors in $\mathbf{Z}^3$:

Lemma 4.8 *Under the algebraic degeneracy condition (A-4) the irreducible equivariant family of projective lines in K^* is exactly generated by the following two "primitive" triplets:*

$$n = k + m, \quad k = a\overline{k}, \quad m = b\overline{m}, \tag{4.8a}$$

$$n = \tilde{k} + \tilde{m}, \quad \tilde{k} = a'\sigma_i(\overline{k}) + b'\sigma_j(\overline{m}), \tag{4.8b}$$

that is,

$$n = a\overline{k} + b\overline{m}, \tag{4.8c}$$

$$n = a'\sigma_i(\overline{k}) + b'\sigma_j(\overline{m}), \tag{4.8d}$$

where $\sigma_i \neq \sigma_j$ are some reflection symmetries, $a,\ b,\ a',\ b'$ are relatively prime integers, positive or negative, and $\overline{k},\ \overline{m}$ are relatively prime integer valued vectors in $\mathbf{Z}^3$, that is:

$$(a, a') = (b, b') = (a, b) = (a', b') = 1, \quad (\overline{k}, \overline{m}) = 1,$$

where (,) denotes the Greatest Common Denominator of two integers. All other resonant wave number triplets are generated by the group actions σ_l, $l = 1, 2, 3$, and homothetic rescalings $(k, m, n) \to \gamma(k, m, n)$, $(\tilde{k}, \tilde{m}, n) \to \gamma(\tilde{k}, \tilde{m}, n)$, $(\gamma \in \mathbf{Z})$ of the "primitive" triplets.

Remark 4.9 *It can be proven that the set of such coupled "primitive" triplets is not empty on the periodic lattice. The algebraic irreducibility condition of Lemma 4.2 implies that $\pm k_3/|k| = \pm\tilde{k}_3/|\tilde{k}|$ and $\pm m_3/|m| = \pm\tilde{m}_3/|\tilde{m}|$, which is obviously verified in equations (4.8).*

Theorem 4.10 *Under conditions of Lemma 4.8 the resonant Euler system reduces to a system of two rigid bodies coupled via $a_n(t)$:*

$$\dot{a}_k = (\lambda_m - \lambda_n)\Gamma a_m a_n \tag{4.9a}$$

$$\dot{a}_m = (\lambda_n - \lambda_k)\Gamma a_n a_k \tag{4.9b}$$

$$\dot{a}_n = (\lambda_k - \lambda_m)\Gamma a_k a_m + (\lambda_{\tilde{k}} - \lambda_{\tilde{m}})\tilde{\Gamma} a_{\tilde{k}} a_{\tilde{m}} \tag{4.9c}$$

$$\dot{a}_{\tilde{m}} = (\lambda_n - \lambda_{\tilde{k}})\tilde{\Gamma} a_n a_{\tilde{k}} \tag{4.9d}$$

$$\dot{a}_{\tilde{k}} = (\lambda_{\tilde{m}} - \lambda_n)\tilde{\Gamma} a_{\tilde{m}} a_n, \tag{4.9e}$$

where $\Gamma = i < \mathbf{\Phi}_k \times \mathbf{\Phi}_m, \mathbf{\Phi}_n^* >$, $\tilde{\Gamma} = i < \mathbf{\Phi}_{\tilde{k}} \times \mathbf{\Phi}_{\tilde{m}}, \mathbf{\Phi}_n^* >$. Energy and Helicity are conserved.

Theorem 4.11 *The resonant system (4.9) possesses three independent conservation laws:*

$$\mathcal{E}_1 = a_k^2 + (1-\alpha)a_m^2, \tag{4.10a}$$
$$\mathcal{E}_2 = a_n^2 + \alpha a_m^2 + (1-\tilde{\alpha})a_{\tilde{m}}^2, \tag{4.10b}$$
$$\mathcal{E}_3 = a_{\tilde{k}}^2 + \tilde{\alpha} a_{\tilde{m}}^2, \tag{4.10c}$$

where

$$\alpha = (\lambda_m - \lambda_k)/(\lambda_n - \lambda_k), \tag{4.11a}$$
$$\tilde{\alpha} = (\lambda_{\tilde{m}} - \lambda_n)/(\lambda_{\tilde{k}} - \lambda_n). \tag{4.11b}$$

Theorem 4.12 *Under the conditions*

$$\lambda_m < \lambda_k < \lambda_n, \tag{4.12a}$$
$$\lambda_{\tilde{m}} < \lambda_n < \lambda_{\tilde{k}}, \tag{4.12b}$$

which imply $\alpha < 0$, $\tilde{\alpha} < 0$, the equilibria $(\pm a_k(0), 0, 0, 0, \pm a_{\tilde{k}}(0))$ are hyperbolic for $|a_{\tilde{k}}(0)|$ small enough with respect to $|a_k(0)|$. The unstable manifolds of these equilibria are one dimensional, and the nonlinear dynamics of system (4.9) are constrained on the ellipse $\mathcal{E}_1$ (4.10a) for $a_k(t)$, $a_m(t)$, the hyperbola $\mathcal{E}_3$ (4.10c) for $a_{\tilde{k}}(t)$, $a_{\tilde{m}}(t)$, and the hyperboloid $\mathcal{E}_2$ (4.10b) for $a_m(t)$, $a_{\tilde{m}}(t)$, $a_n(t)$.

Theorem 4.13 *Let the 2-manifold $\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3$ be coordinatized by $(a_m, a_{\tilde{m}})$. On this 2-manifold, the resonant system (4.9) is Hamiltonian, and therefore integrable. Its Hamiltonian vector field $\mathbf{h}$ is defined by*

$$\iota_{\mathbf{h}}\omega \;=\; \Gamma(\lambda_n - \lambda_k)\frac{da_{\tilde{m}}}{a_{\tilde{k}}} - \tilde{\Gamma}(\lambda_n - \lambda_{\tilde{k}})\frac{da_m}{a_k}, \tag{4.13}$$

where $\iota_{\mathbf{h}}\omega$ designates the inner product of the symplectic 2-form

$$\omega = \frac{da_m \wedge da_{\tilde{m}}}{a_k a_n a_{\tilde{k}}} \tag{4.14}$$

with the vector field $\mathbf{h}$.

Proof of Theorem 4.13: Eliminating $a_k(t)$ via $\mathcal{E}_1$, $a_n(t)$ via $\mathcal{E}_2$, $a_{\tilde{k}}(t)$ via $\mathcal{E}_3$, the resonant system (4.9) reduces to:

$$\begin{aligned}
\dot{a}_m &= \pm\Gamma(\lambda_n - \lambda_k)(\mathcal{E}_1 - (1-\alpha)a_m^2)^{\frac{1}{2}}(\mathcal{E}_2 - \alpha a_m^2 + (\tilde{\alpha}-1)a_{\tilde{m}}^2)^{\frac{1}{2}} \\
\dot{a}_{\tilde{m}} &= \pm\tilde{\Gamma}(\lambda_n - \lambda_{\tilde{k}})(\mathcal{E}_2 - \alpha a_m^2 + (\tilde{\alpha}-1)a_{\tilde{m}}^2)^{\frac{1}{2}}(\mathcal{E}_3 - \tilde{\alpha}a_{\tilde{m}}^2)^{\frac{1}{2}};
\end{aligned}$$

after changing the time variable into

$$t \to \int_0^t (\mathcal{E}_1 - (1-\alpha)a_m^2)^{\frac{1}{2}}(\mathcal{E}_2 - \alpha a_m^2 + (\tilde{\alpha}-1)a_{\tilde{m}}^2)^{\frac{1}{2}}(\mathcal{E}_3 - \tilde{\alpha}a_{\tilde{m}}^2)^{\frac{1}{2}}\, ds\,.$$

On each component of the manifold $\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3$, the following functionals are conserved:

$$\begin{aligned}\mathcal{H}(a_m, a_{\tilde{m}}) = \quad & \pm \quad \tilde{\Gamma}(\lambda_n - \lambda_{\tilde{k}}) \int \frac{da_m}{(\mathcal{E}_1 - (1-\alpha)a_m^2)^{1/2}} \\ & \pm \quad \Gamma(\lambda_n - \lambda_k) \int \frac{da_{\tilde{m}}}{(\mathcal{E}_3 - \tilde{\alpha} a_{\tilde{m}}^2)^{1/2}}. \qquad \blacksquare\end{aligned}$$

Observe that the system of two coupled rigid bodies (4.9) does not seem to admit a simple Lie-Poisson bracket in the original variables $(a_k, a_m, a_n, a_{\tilde{m}}, a_{\tilde{k}})$. Yet, when restricted to the 2-manifold $\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3$ that is invariant under the flow of (4.9), it is Hamiltonian and therefore integrable.

This raises the following interesting issue: according to the shadowing Theorem 2.10, the Euler dynamics remains asymptotically close to that of chains of coupled $SO(3;\mathbf{R})$ and $SO(3;\mathbf{C})$ rigid body systems. Perhaps some new information could be obtained in this way. We are currently investigating this question and will report on it in a forthcoming publication [G-M-N].

Already the simple 5-dimensional system (4.9) has interesting dynamical properties, wich we could not find in the existing literature on systems related to spinning tops.

Consider for instance the dynamics of the resonant system (4.9) with I.C. topologically close to the hyperbola equilibria $(\pm a_k(0), 0, 0, 0, \pm a_{\tilde{k}}(0))$. Under the conditions of (4.12) and with the help of the integrability Theorem 4.13, it is easy to construct equivariant families of homoclinic cycles at these hyperbolic critical points:

Corollary 4.14 *The hyperbolic critical points* $(\pm a_k(0), 0, 0, 0, \pm a_{\tilde{k}}(0))$ *possess 1-dimensional homoclinic cycles on the cones*

$$a_n^2 + (1-\tilde{\alpha})a_{\tilde{m}}^2 = -\alpha a_m^2 \tag{4.15}$$

with $\alpha < 0,\ \tilde{\alpha} < 0$.

Note that these are genuine homoclinic cycles, NOT sums of heteroclinic connections. Initial conditions for the resonant system (4.9) are now chosen in a small neighborhood of these hyperbolic critical points, the corresponding orbits are topologically close to these cycles. With the ordering:

$$\lambda_m < \lambda_k < \lambda_n, \tag{4.16a}$$
$$|\lambda_k| \ll |\lambda_m|,\ |\lambda_k| \ll \lambda_n, \tag{4.16b}$$
$$\lambda_{\tilde{m}} < \lambda_n < \lambda_{\tilde{k}}, \tag{4.16c}$$
$$|\lambda_{\tilde{m}}| \ll \lambda_{\tilde{k}}, \tag{4.16d}$$
$$\lambda_{\tilde{k}} \gg \lambda_n, \tag{4.16e}$$

which can be realized with $|\frac{a'}{a}| \gg 1$ and $|\frac{b'}{b}| \ll 1$ in the resonant triplets (4.8), we can demonstrate bursting dynamics akin to Theorem 3.9 and 3.11 for enstrophy and $\mathbf{H}^s$ norms, $s \geq 2$. The interesting feature is the maximization of $|a_{\tilde{k}}(t)|$ near the turning points of the homoclinic cycles on the cones (4.15). This corresponds to transfer of energy to the smallest scale $\tilde{k}$, $\lambda_{\tilde{k}}$.

In a publication in preparation, we investigate infinite systems of the coupled rigid bodies equations (4.9).

APPENDIX

We focus on a resonant wave number triplet $(n, k, m) \in (\mathbf{Z}^*)^3$ verifying

- the convolution relation
$$n = k + m, \tag{A-1}$$
- the resonant 3-wave resonance relation
$$\pm \frac{n_3}{\sqrt{\vartheta_1 n_1^2 + \vartheta_2 n_2^2 + \vartheta_3 n_3^2}} \pm \frac{k_3}{\sqrt{\vartheta_1 k_1^2 + \vartheta_2 k_2^2 + \vartheta_3 k_3^2}} \pm \frac{m_3}{\sqrt{\vartheta_1 m_1^2 + \vartheta_2 m_2^2 + \vartheta_3 m_3^2}} = 0, \tag{A-2}$$
- the condition of "non-catalyticity"
$$k_3 m_3 n_3 \neq 0, \tag{A-3}$$
- and the degeneracy condition of [B-M-N4] (see p26)
$$G^{ir}_{i,j}(k, m) = k_i n_j m_l + k_l m_j n_i = 0, \tag{A-4}$$
where (i, j, l) is a permutation of $(1, 2, 3)$.

Then, we know (see lemma 3.5 (2) of [B-M-N4]) that the system of equations (A-3)-(A-4) for the unknown k and m, given the vector n, admits exactly 4 solutions in $\mathbf{Z}^3 \times \mathbf{Z}^3$:

$$(k, m), \quad (m, k), \quad (\tilde{k}, \tilde{m}), \quad (\tilde{m}, \tilde{k}).$$

Here k and m are the two vectors of the original resonant triplet, whereas

$$\tilde{k} = \alpha \sigma_i(k), \quad \tilde{m} = \beta \sigma_j(m)$$

where

$$\alpha = \frac{m_i k_l - m_l k_i}{m_i k_l + m_l k_i} \notin \{0, \pm 1\} \text{ and } \beta = \frac{m_l k_j - m_j k_l}{m_l k_j + m_j k_l} \notin \{0, \pm 1\}$$

and where the symmetries σ_i and σ_j are defined by

$$\sigma_i : u = (u_l)_{l=1,2,3} \to \left((-1)^{\delta_{il}} u_l\right)_{l=1,2,3}.$$

One verifies that

$$\sigma_i^2 = \sigma_j^2 = Id, \quad \sigma_i\sigma_j = \sigma_j\sigma_i = -\sigma_l.$$

That is, the group generated by σ_i and σ_j is the Klein group $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$.

Let us first write the irrational numbers α and β under the irreducible representation

$$\alpha = \frac{a'}{a}, \quad \beta = \frac{b'}{b}, \quad \text{with } a, a', b, b' \in \mathbf{Z}^* \text{ and } (a, a') = (b, b') = 1,$$

where $(\ ,\ )$ denotes the Greatest Common Denominator of the integer pair. From $\tilde{k} \in \mathbf{Z}^3$, it follows that $a|a'k$; but since $(a, a') = 1$, the Euclid's lemma yields that $a|k$. Similarly, $b|m$. Now set

$$\overline{k} = \frac{1}{a}k \in \mathbf{Z}^3, \quad \overline{m} = \frac{1}{b}m \in \mathbf{Z}^3.$$

Hence the integer vector n admits the two decompositions

$$n = a\overline{k} + b\overline{m} = a'\sigma_i(\overline{k}) + b'\sigma_j(\overline{m}).$$

Since the function

$$z \longmapsto \frac{z_3}{\sqrt{\vartheta_1 z_1^2 + \vartheta_2 z_2^2 + \vartheta_3 z_3^2}}$$

is homogeneous of degree 0, we see that within the resonance condition (A-2) we can replace each vector k, m and n by any colinear vectors - either integer or not. Suppose now that there exists some positive integer $d \neq 1$ such that $d|\overline{k}$; then $d|n$, so that by setting

$$n_0 = \frac{1}{d}n, \quad k_0 = \frac{1}{d}\overline{k}, \quad m_0 = \frac{1}{d}\overline{m}$$

we finally obtain

$$n_0 = ak_0 + bk_0 = a'\sigma_i(k_0) + b'\sigma_j(m_0).$$

The triplets (n_0, ak_0, bm_0) and $(n_0, a'\sigma_i(k_0), b'\sigma_j(m_0))$ further verify from the above remark, the convolution relation (A-1) and the resonance relation (A-2). Hence, without loss of generality, we can assume that the only positive integer d such that $d|\overline{k}$ and $d|\overline{m}$ is 1; which we denote by

$$(\overline{k}, \overline{m}) = 1.$$

Equivalently,

$$\overline{k}_1\mathbf{Z} + \overline{k}_2\mathbf{Z} + \overline{k}_3\mathbf{Z} + \overline{m}_1\mathbf{Z} + \overline{m}_2\mathbf{Z} + \overline{m}_3\mathbf{Z} + = \mathbf{Z}.$$

Finally, suppose there exists some positive integer $d \neq 1$ such that $d|a$ and $d|b$. Then $d|n$; set

$$n_0 = \frac{1}{d}n, \quad a_0 = \frac{1}{d}a, \quad b_0 = \frac{1}{d}b.$$

Observe that

$$G^{ir}_{i,j}(a_0\overline{k}, b_0\overline{m}) = \frac{1}{d^3}G^{ir}_{i,j}(a\overline{k}, b\overline{m}) = 0.$$

It follows from lemma 3.5 (2) of [B-M-N4] that the vector n_0 of the resonant triplet $(n_0, a_0\overline{k}, b_0\overline{m})$ can also be written as

$$n_0 = \hat{k} + \hat{m} \text{ with } (n_0, \hat{k}, \hat{m}) \text{ verifying (A-2).}$$

But then

$$n = dn_0 = a\overline{k} + b\overline{m} = a'\sigma_i(\overline{k}) + b'\sigma_j(\overline{m}) = d\hat{k} + d\hat{m}.$$

From lemma 3.5 (2) of [B-M-N4], $(d\hat{k}, d\hat{m})$ must coincide with either one of the pairs

$$(a'\sigma_i(\overline{k}), b'\sigma_j(\overline{m})), \quad (b'\sigma_j(\overline{m}), a'\sigma_i(\overline{k})).$$

In particular, $d|a'\overline{k}$ and $d|b'\overline{m}$. Since $d|a$ and $(a, a') = 1$, we have (d, a'); similarly $(d, b') = 1$. But then Euclid's lemma yields that $d|\overline{k}$ and $d|\overline{m}$, which contradicts the fact that $(\overline{k}, \overline{m}) = 1$. Hence we have proven that $(a, b) = 1$. In a similar way, one can show that $(a', b') = 1$.

Conclusion: It follows from the above study that $n \in \mathbf{Z}^*$ admits the two decompositions

$$n = a\overline{k} + b\overline{m} = a'\sigma_i(\overline{k}) + b'\sigma_j(\overline{m})$$

with

$$(a, a') = (b, b') = (a, b) = (a', b') = 1, \quad (\overline{k}, \overline{m}) = 1.$$

The triplets $(n, a\overline{k}, b\overline{m})$ and $(n, a'\sigma_i(\overline{k}), b'\sigma_j(\overline{m}))$ both verify the resonant condition (A-2) (from the homogeneity of this condition) as well as the condition of non-catalyticity (A-3). Indeed, $aba'b' \neq 0$ and the condition (A-3) on the initial triplet (n, k, m) imply that the reduced triplet $(n, \overline{k}, \overline{m})$ also verifies (A-3)). Finally, the degeneracy condition (A-4)

$$G^{ir}_{i,j}(a\overline{k}, b\overline{m}) = 0$$

is verified.

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